On Identification Issues in Business Cycle Accounting Models*

Pedro Brinca† Nikolay Iskrev‡ Francesca Loria§

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Abstract

Since its introduction by Chari et al. (2007), Business Cycle Accounting (BCA) exercises have become widespread. Much attention has been devoted to the results of such exercises and to methodological departures from the baseline methodology. Little attention has been paid to identification issues within these classes of models.In this paper we investigate whether such issues are of concern in the original methodology and in an extension proposed by Šustek (2011) called Monetary BCA. We resort to two types of identification tests in population. One concerns strict identification as theorized by Komunjer and Ng (2011) while the other deals both with strict and weak identification as in Iskrev (2015). Most importantly, we explore the extent to which these weak identification problems affect the main economic takeaways and find that the identification deficiencies are not relevant for the standard BCA model. Finally, we compute some statistics of interest to practitioners of the BCA methodology.

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†Nova School of Business and Economics, Universidade Nova de Lisboa, Campus de Campolide, 1099-032, Lisboa, Portugal. E-mail: pedro.brinca@novasbe.pt.

‡Banco de Portugal, Av. Almirante Reis 71, 1150-012, Lisboa, Portugal. E-mail: niskrev@bportugal.pt.

§Corresponding Author. Board of Governors of the Federal Reserve System,20th St & Constitution Ave NW, Washington, D.C. 20551, U.S.A. E-mail: francescaloria.work@gmail.com.

1 Introduction

The Business Cycle Accounting (BCA) procedure developed by Chari et al. (2007) and recently revived by Brinca et al. (2016) has sparked great interest among quantitative and theoretical economists. This tool allows to look at the data through the lens of a standard business cycle model and, most importantly, to detect and quantitatively assess to which extent and in which equilibrium conditions the model performs better or worse. The so-called "wedges" - which in practical terms are whatever makes the equilibrium conditions not hold - can be mapped into frictions of richer models and viceversa. In this context, the BCA exercise can thus be thought of as some ex-ante diagnosis tool for researchers to know in which broad classes of models it is worth or not worth investing time if one wants to explain fluctuations in macroeconomic aggregates such as GDP, investment and working hours during a particular economic episode.

Business Cycle Accounting has become a standard tool of business cycle analysis and, since its inception, literally hundreds of applications of the original methodology have been performed.

Examples can be found in Kobayashi and Inaba (2006) for Japan, Simonovska and Söderling (2008) for Chile and Lamas (2009) for Argentina, Mexico and Brazil. The results seem to conclude, much in line with Chari et al. (2007), that total factor productivity and distortions to the labor choice are relevant, whereas distortions to the savings decision are considerably less important. Some authors focus their analysis to one type of distortions as in Restrepo-Echavarria and Cheremukhin (2010) or Cociuba and Ueberfedt (2010), where the focus is on the labor-leisure margin, or other numerous studies which deal with total factor productivity such as Islam et al. (2006). Another line of work looks into a selected sample of countries and into specific periods of fluctuations such as output drops (see, e.g., Dooyeon and Doblas-Madrid (2012)). Brinca (2014) instead provides a comprehensive exercise for 22 OECD countries covering the period from 1970 to 2012. It looks at the quantitative relevance of each distortion over the whole business cycle and not just booms or busts. The results confirm the findings of Chari et al. (2007) regarding the Great Depression and the 1981 recession in the U.S., while stressing the relevance of the international channels of transmission of these distortions.

In terms of methodological departures, Otsu (2009) extends the methodology to a two country setting. Šustek (2011) includes a Taylor-type nominal interest rate setting rule and an extra asset, government bonds, to study the behavior of nominal variables such as the nominal interest rate on bonds and the inflation rate, and gives it the name of Monetary Business Cycle Accounting (MBCA). These departures have also been explored. For instance, Brinca (2013) applies Šustek (2011) model to perform a Monetary Business Cycle Accounting exercise for Sweden, comparing the 1990's crisis with the period of the Great Recession.

While much attention has been devoted to the Business Cycle Accounting methodology, no efforts have been made in investigating whether the parameter estimation procedure associated with this tool is affected by identification deficiencies. The question of identifiability in dynamic stochastic general equilibrium (DSGE) is an important one as it might jeopardize the consistency and adversely affect the precision of parameter estimates. This issue becomes even more important in light of the fact that, in the past years, DSGE models have become a standard and important asset within the toolkit of economic policymakers to make quantitative statements about real and nominal variables. When these models are brought to the data researchers should be cautious in taking for granted the empirical credibility of their estimated parameters and, thus, of the economic implications that the latter entail. Indeed, due to identifiability issues inherent to DSGE models, it is far from obvious that parameters can be inferred successfully even when one has an infinite sample of observed data and when full-information methods such as maximum likelihood are employed in estimation. Most importantly, as pointed out by Canova and Sala (2009), in some cases these identification deficiencies can result in significantly different economic inference from the theoretical models of interest. To the extent that the Business Cycle Accounting exercises by Chari et al. (2007) and Šustek (2011) draw quantitative conclusions from their respective DSGE models of reference it is of outmost importance that their identification potential is carefully analyzed.

This paper builds up on the literature which studies local sample and population identification issues specific to linearized DSGE models. Our methodological approach to the analysis of such issues is closely related to the work by Canova and Sala (2009) whose contribution was to provide (i) a working language which allows researchers to classify identification problems, (ii) formal graphical inspection tools to detect those problems and (iii) possible ways to obviate them. Due to the high number of parameters to be estimated in both models graphical inspection quickly becomes unmanageable and dispersive. This leads us to bring into our toolkit the formal identification tests developed by Komunjer and Ng (2011) and Iskrev (2015).

The results presented in this paper suggest that the standard and monetary BCA model do not suffer from strict identification failures when estimation is restricted to the parameters governing the law of motion of the latent variables and that this is not true anymore once one extends the estimation to the deep parameters of the model. We show that restricting estimation of some deep parameters can be obviate these strict identification failures. We also find that both models are affected by weak identification deficiencies and that these are induced by several parameters of the model not exerting a distinct effect on our objective function of interest, namely the likelihood function of the model. Finally, we explore to which extent this type of identification failure affects the main conclusions to be drawn from (M)BCA exercises. We find that the main takeaways from a standard BCA exercise are not overturned when one explicitly takes into account the weak identifiability of the model's parameters. We are still investigating whether this result holds through in the monetary BCA framework.

Finally, we analyze how overall identification strength of the estimated parameter vector varies across sample sizes. To do so, we compute empirical distance measures as in Qu and Tkachenko (2016). We find that in both models the parameter set is overall well identified even if a practitioner is faced with a data set of only 20 data points per observable.

The paper is organized as follows. Section 2 presents the prototype MBCA economy. The methodology to run the identification tests is illustrated in Section 3. Their results are reported in Section 4 while Section 5 discusses their economic relevance. In Section 6 we present statistics of interest to practitioners and then conclude in Section 7.

2 The Prototype (M)BCA Economy

We start by introducing the MBCA prototype economy as in Šustek (2011), which is an extension of the prototype economy in Chari et al. (2007), a neoclassical growth model with labor-leisure choice. The extension consists in allowing for an extra asset (government bonds) and introducing a nominal interest rate setting rule.

2.1 Description of the Economy

There is an infinitely-lived representative agent that maximizes expected discounted utility and a representative firm, both price-takers in all markets. The economy experiences one of finitely many events s_t, where $s^t = (s_0, ..., s_t)$ is the history of events up to period t which occur with probability $\pi_t(s^t)$. There are six exogenous stochastic variables which are all function of the random variable s^t. Four of them are the same as in Chari et al. (2007). These are the efficiency wedge $Z_t(s^t)$, the labor wedge $1 - \tau_{l,t}(s^t)$, the investment wedge $1/[1+\tau_{x,t}(s^t)]$ and the government wedge $g_t(s^t)$. With the Šustek (2011) extension, two more stochastic variables are added: an asset market wedge $1/[1 + \tau_{b,t}(s^t)]$ and a monetary policy wedge $\tilde{R}_t(s^t)$.

The representative household chooses how much to consume $c_t(s^t)$ and how much labor to supply $l_t(s^t)$. Given the discount factor β it solves the following maximization problem:

$$\max_{\{c_t(s^t), l_t(s^t), k_{t+1}(s^t), b_t(s^t)\}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t(s^t), 1 - l(s^t)\right)(1+g_n)^t, \tag{2.1}$$

subject to the budget constraint:

$$c_t(s^t) + [1+\tau_x(s^t)]x_t(s^t) + [1+\tau_b(s^t)]\left[(1+g_n)\frac{b_t(s^t)}{[1+R_t(s^t)]p_t(s^t)} - \frac{b_{t-1}(s^{t-1})}{p_t(s^t)}\right]$$
$$= [1-\tau_l(s^t)]w_t(s^t)l_t(s^t) + r_t(s^t)k_t(s^{t-1}) + T_t(s^t) \tag{2.2}$$

where $x_t(s^t)$ is investment, g_n the population growth rate, $b_t(s^t)$ are bond holdings paying a net nominal rate of return $R_t(s^t)$ in all states of the world, $p_t(s^t)$ is the nominal price of goods in terms of a numeraire, $w_t(s^t)$ the wage rate, $r_t(s^t)$ the real rental rate of return on capital $k_t(s^t)$ held at the beginning of period and $T_t(s^t)$ lump-sum transfers from the government. Capital accumulation follows

$$(1+g_n)k_{t+1}(s^t) = (1-\delta)k_t(s^t) + x_t(s^t) \tag{2.3}$$

where δ is capital's depreciation rate. The production function for the representative firm is given by

$$y_t(s^t) = F\left(k_t(s^{t-1}), Z_t(s^t)l_t(s^t)\right), \tag{2.4}$$

which is assumed to exhibit constant returns to scale. The aggregate resource constraint is then given by

$$y_t(s^t) = c_t(s^t) + g_t(s^t) + x_t(s^t). \tag{2.5}$$

There is a monetary authority who reacts to deviations from steady-state output y and inflation π by setting the nominal interest rate $R_t(s^t)$ according to

$$R_t(s^t) = (1-\rho_R)\left[R + \omega_y(\ln y_t(s^t) - \ln y) + \omega_\pi(\pi_t(s^t) - \pi)\right] + \rho_R R_{t-1}(s^{t-1}) + \tilde{R}_t(s^t) \tag{2.6}$$

where $\rho_R \in [0,1)$ and $\pi_t(s^t) \equiv \ln\, p_t(s^t) - \ln\, p_{t-1}(s^{t-1})$ is the inflation rate. In addition, it is assumed that $\omega_\pi > 1$, thus eliminating explosive paths for inflation.

Just like in Chari et al. (2007) it is assumed that the state s_t follows a Makov process of the type

$$s_{t+1} = P_0 + P s_t + Q\varepsilon_{s,t+1}, \tag{2.7}$$

where $\varepsilon_{s,t+1} \sim N(0, I)$. Moreover, the mapping between this process of the underlying event $s_t = (s_{Zt}, s_{lt}, s_{xt}, s_{gt}, s_{bt}, s_{\tilde{R}t})$ and the wedges $\left(Z_t, 1-\tau_{l,t}, \frac{1}{1+\tau_{x,t}}, g_t, \frac{1}{1+\tau_{b,t}}, \tilde{R}_t\right)$ is one-to-one and onto. This setup is equivalent to assuming that agents use only past realizations of wedges to forecast future ones. Note that in the case where the matrix P is diagonal then, irrespectively of whether the covariance matrix QQ' is diagonal or not (i.e., whether the shocks are allowed to be correlated or not), the Monetary BCA model is block-recursive in the sense that shocks to the wedges of the standard BCA setup only affect real variables while leaving the model's nominal variables - interest rate R_t and inflation rate π_t - unaffected. In this sense we can think of the MBCA theoretical framework as nesting the plain-vanilla BCA one. This is why we avoid a detailed exposition of the latter.

2.2 Equilibrium Conditions

Equilibrium allocations are pinned down by the production function in (2.4), the aggregate resource constraint in (2.5) and the first order conditions with respect to labor, capital and bond holdings below:

$$-\frac{U_{l,t}(s^t)}{U_{c,t}(s^t)} = [1-\tau_{l,t}(s^t)]w_t(s^t), \tag{2.8}$$

$$1 = \beta\ \mathbb{E}_t\left\{\frac{U_{c,t+1}(s^{t+1})}{U_{c,t}(s^t)}\left[\frac{[1+\tau_{x,t+1}(s^{t+1})](1-\delta)+r_{t+1}(s^{t+1})}{1+\tau_{x,t}(s^t)}\right]\right\}, \tag{2.9}$$

$$1 = \beta\ \mathbb{E}_t\left\{\frac{U_{c,t+1}(s^{t+1})}{U_{c,t}(s^t)}\frac{1+\tau_{b,t+1}(s^{t+1})}{1+\tau_{b,t}(s^t)}\frac{p_t(s^t)}{p_{t+1}(s^{t+1})}[1+R_t(s^t)]\right\}. \tag{2.10}$$

In Appendix B we derive these conditions and present the full-fledged model.

The notation $\hat{v} \equiv \frac{v_t}{(1+g_z)^t} \equiv \frac{V_t}{N_t(1+g_z)^t}$ refers to model variables V_t which are not only expressed in per-capita terms v_t but also detrended $\hat{v}_t$.

2.3 Operational Model

In the operational version of the model which we bring to the data we consider quantities which are not only expressed in per-capita terms but also detrended (see Appendix B.5 for derivations). To highlight the differences between this model's and the previous model's variables we introduce the notation $\left(\hat{v} \equiv \frac{v_t}{(1+g_z)^t} \equiv \frac{V_t}{N_t(1+g_z)^t}\right)$.

The model is given by the CRS Production Function

$$\hat{y}_t(s^t) = \hat{k}_t(s^{t-1})^\alpha \left(z_t l_t(s^t)\right)^{1-\alpha}, \tag{2.11}$$

the aggregate resource constraint

$$\hat{y}_t(s^t) = \hat{c}_t(s^t) + \hat{g}_t + \hat{x}_t(s^t), \tag{2.12}$$

the capital accumulation law

$$(1+g_n)(1+g_z)\hat{k}_{t+1}(z^t) = (1-\delta)\hat{k}_t(z^{t-1}) + \hat{x}_t(z^t), \tag{2.13}$$

the Taylor rule

$$R_t(s^t) = (1-\rho_R)\left[R + \omega_y(\ln\,\hat{y}_t(s^t) - \ln\,\hat{y}) + \omega_\pi(\pi_t(s^t)-\pi)\right] + \rho_R R_{t-1}(s^{t-1}) + \tilde{R}_t, \tag{2.14}$$

the F.O.C. for labor

$$\psi\frac{\hat{c}_t(s^t)}{1-l_t(s^t)} = (1-\tau_{l,t})(1-\alpha)\hat{k}_t(s^{t-1})^\alpha z_t^{1-\alpha} l_t(s^t)^{-\alpha}, \tag{2.15}$$

the F.O.C. for capital

$$1 = \tilde{\beta}\mathbb{E}_t\left\{\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{-\sigma}\left(\frac{1-l_{t+1}}{1-l_t}\right)^{\psi(1-\sigma)}\left[\frac{(1+\tau_{x,t+1})(1-\delta)+\alpha\hat{k}_{t+1}(s^t)^{\alpha-1}\left(z_{t+1}l_{t+1}(s^{t+1})\right)^{1-\alpha}}{1+\tau_{x,t}}\right]\right\}, \tag{2.16}$$

and the F.O.C. Bonds

$$1 = \tilde{\beta}\mathbb{E}_t\left\{\frac{\hat{c}_t(s^t)}{\hat{c}_{t+1}(s^{t+1})}\frac{1+\tau_{b,t+1}}{1+\tau_{b,t}}\frac{p_t(s^t)}{p_{t+1}(s^{t+1})}[1+R_t(s^t)]\right\}, \tag{2.17}$$

where $\tilde{\beta} = \beta/(1+g_z)$.

Notice that the operational model the efficiency wedge is thus given by $z_t = \frac{Z_t}{(1+g_n)^t}$ and the government wedge by $\hat{g}_t = \frac{g_t}{(1+g_n)^t}$.

The structural parameters of the BCA model are (i) g_n, population growth rate of labor-augmenting technological process, (ii) g_z, growth rate of labor-augmenting technological process, (iii) α, parameter which determines the share of (and weight on) net capital stock in the Cobb-Douglas CRS production function, (iv) β, subjective discount factor, reflecting the time preference of the household, (v) δ, depreciation rate of net capital stock and (vi) ψ, Frisch elasticity of labor supply. In the MBCA model the deep parameters also include (vii) ρ_R, weight on lagged nominal interest rate in the Taylor rule (extent of "interest rate smoothing"), (viii) ω_π, coefficient on deviations of inflation from its steady state value in Taylor rule, (ix) ω_y, coefficient on deviations of output from its steady state value in Taylor rule and (x) π_{ss}, steady state inflation.

3 Methodology

The problem of identification emerges when a researcher seeks to infer the parameters of his theoretical model from a sample of observed data. In general and loosely speaking, the requirements for "successful estimation" are that (i) the objective function (e.g., the log-likelihood function) has a unique maximum, (ii) the Hessian at the mode is negative definite and has full rank and that (iii) the objective function exhibits "sufficient" curvature. Local identifiability is crucial as it guarantees consistent parameter estimation and that the estimator has the usual asymptotic properties, where by "local" we mean that the maximum in condition (i) above is at least locally unique. It is thus of outmost importance to investigate thoroughly whether the conditions for local identifiability are satisfied within the standard and monetary Business Cycle Accounting framework.

In our analysis we employ strict and weak identification tests. Both are performed in population, i.e., given a theoretical sample of infinite length. The former provides a yes or no answer to the question whether a parameter is identifiable or not and, thus, whether condition (ii) above is satisfied or not. The latter investigates condition (iii) above and is thus informative about whether the likelihood exhibits small curvature, where small is in relation to an economically relevant parameter range. The weak identification test also instructs about whether the small curvature is due to the fact that the parameters have no effect on the objective function or to the fact that the variation which they induce on the objective function is not distinct enough from other parameters which are being estimated.

As to the strict identification tests in *population*, consider the case where the researcher has a sample of length T generated by (3.4) with $\theta = \theta_0$. In this context, one can ask the following question: If the sample was infinitely large, i.e., $T \to \infty$, under which conditions would it be possible to uncover the value θ_0 and the model that generated the data? Problems specific to the dynamic nature of DSGE models make it difficult to test whether the conditions for local identifiability mentioned above hold given a sample of data. Indeed, as pointed out by Canova and Sala (2009), the mapping from the structural parameters to the solution coefficients is typically unknown and the latter, in turn, usually appear in a nonlinear way in the objective function. These are some of the reasons why the rank and order conditions of Rothenberg (1971) derived for simultaneous system of equations cannot be applied. As pointed out by Komunjer and Ng (2011) other reasons are that classical conditions (i) are derived for static models whose reduced form errors are orthogonal to the regressors, an assumption which is implausible for DSGE models given that they are dynamic, (ii) do not recognize the fact that reduced form parameters themselves can be not identifiable unless all state variables are observed, (iii) are for simultaneous system of equations which is not the form of DSGE model solutions.

Alternative rank and order conditions for strict identification are derived by Komunjer and Ng (2011) using the spectral density matrix and by Iskrev (2015) using the likelihood function of the DSGE model. Both tests treat parameter identification as a property of the underlying structural model. This is motivated by the fact that DSGE models completely characterize the data generating process. This is in contrast with other types of models where the mapping from the model to the data is only partially known. Therefore, the economic model is the origin of identification problems which appear in a particular data set. It is then straightforward to see that identification problems may occur as an intrinsic property of the model when, for instance, the restrictions that the model imposes on the joint distribution of the observed variables do not contain sufficient information about some parameters of interest. It is important to recognize the fact that, in general, these restrictions are a function of the parameters. Hence, also the data crucially contributes to identify those parameter values for which the model can account well for the movements in the data.

An advantage of the Iskrev (2015) approach over Komunjer and Ng (2011) is that key objects of interest are obtained analytically rather than numerically. Furthermore, it can detect not only strict but also weak identification problems. A central tool in his analysis is the expected Fisher information matrix, as first suggested by Rothenberg (1971). It is intuitive to understand why this matrix comes handy to study identification problems. The information matrix measures the curvature of the expected log-likelihood surface and, as pointed out by Rothenberg (1971), it is informative about the (degree of) informational content available in the sample about the unknown parameters. For instance, one should expect identification deficiencies to arise when the log-likelihood surface is flat or nearly flat with respect to the parameters to be estimated. The degree of "flatness" can be detected and quantified via the information matrix.

There are two main reasons why parameters might be unidentifiable or just weakly identifiable; they can be broadly classified as a "sensitivity" and a "collinearity" factor. They originate from the fact that the economic features which operate via the problematic parameters may be nearly or completely irrelevant with respect to the variables of the model used in estimation. The "sensitivity" factor signals that the identification problem occurs because the features are inherently unimportant while the "collinearity" factor attributes it to the nearly redundant nature of these features given others present in the model. The information matrix can be used to assess the importance of the two factors via a simple decomposition. This analysis thus allows not only to flag problematic parameters but also to quantify the strength and discern the nature of their identification deficiencies.

3.1 State Space Form

Following Chari et al. (2007) the state space form of the model is given by

$$\begin{aligned} X_{t+1} &= A(\theta)X_t + B(\theta)\varepsilon_{t+1}, \\ Y_t &= \widehat{C}(\theta)X_t + \omega_t, \\ \omega_t &= \widehat{D}(\theta)\omega_{t-1} + \eta_t, \end{aligned} \tag{3.1}$$

where, in the standard BCA model, the vectors of state variables and the vector of observables are respectively given by $X_t = [\log(\hat{k}_t) - \log(k),\ \log(z_t) - \log(z), \tau_{lt} - \tau_l, \tau_{xt} - \tau_x, \log(\hat{g}_t) - \log(\hat{g}), 1]'$ $Y_t = [\log \hat{y}_t - \log(\hat{y}), \log \hat{x}_t - \log(\hat{x}),\ \log l_t - \log(l),\ \log \hat{g}_t - \log(\hat{g})]'$, whereas in the monetary BCA model $X_t = [\log(\hat{k}_t) - \log(k),\ \log(z_t) - \log(z), \tau_{lt} - \tau_l, \tau_{xt} - \tau_x, \log(\hat{g}_t) - \log(\hat{g}), \tau_{bt} - \tau_b, \tilde{R}_t - \tilde{R}, 1]'$, $Y_t = [\log \hat{y}_t - \log(\hat{y}),\ \log \hat{x}_t - \log(\hat{x}),\ \log l_t - \log(l),\ \log \hat{g}_t - \log(\hat{y}),\ R_t - R,\ \pi_t - \pi]'$.

The parameters of the model are collected in a vector θ and belong to a set $\Theta \subseteq \mathbb{R}^{n_\theta}$. The matrices A, B, $\widehat{C}$ and $\widehat{D}$ are respectively the ones describing (i) the transition of the states, (ii) the variance covariance matrix of the shocks to the wedges $u_t \equiv B(\theta)\varepsilon_t$ given by $\Omega \equiv BB'$ since it is assumed that $\mathbb{E}[\varepsilon_t \varepsilon_t'] = I$, (iii) the mapping from the states to the observables and (iv) the serial correlation of the measurement errors (set to zero here).

For both models, the state space matrices $A(\theta)$ and $B(\theta)$ are obtained by solving the rational expectations systems using the Gensys algorithm developed by Sims (2002) (see Appendix C for more details).

Let us assume that $\mathbb{E}\eta_t\eta_t' = R$ and $\mathbb{E}\varepsilon_t\eta_s' = 0$ for all periods t and s. Next, define $\bar{Y}_t \equiv Y_{t+1} - \widehat{D}Y_t$. Then we can rewrite (3.1) as

$$\begin{aligned} X_{t+1} &= A(\theta)X_t + B(\theta)\varepsilon_{t+1}, \\ \bar{Y}_t &= \bar{C}(\theta)X_t + \widehat{C}(\theta)B(\theta)\varepsilon_{t+1} + \eta_{t+1}, \end{aligned} \tag{3.2}$$

where $\bar{C}(\theta) = \widehat{C}(\theta)A(\theta) - \widehat{D}(\theta)\widehat{C}(\theta)$.

Stacking the vector of innovations and measurement errors into a $n_\epsilon \times 1$ vector $\epsilon_t = (\varepsilon_t', \eta_t')$ yields the following representation

$$\begin{aligned} X_{t+1} &= A(\theta)X_t + B(\theta)\varepsilon_{t+1}, \\ \bar{Y}_t &= \bar{C}(\theta)X_t + \bar{D}(\theta)\epsilon_{t+1}. \end{aligned} \tag{3.3}$$

In all periods and all identification tests we set the measurement errors equal to zero so that $\left(\widehat{D} = R = 0_{4\times 4}\right)$ and

$$\begin{aligned} \underbrace{X_{t+1}}_{n_X\times 1} &= \underbrace{A(\theta)}_{n_X\times n_X}\underbrace{X_t}_{n_X\times 1} + \underbrace{B(\theta)}_{n_X\times n_\varepsilon}\underbrace{\varepsilon_{t+1}}_{n_\varepsilon\times 1}, \\ \underbrace{Y_{t+1}}_{n_Y\times 1} &= \underbrace{C(\theta)}_{n_Y\times n_X}\underbrace{X_t}_{n_X\times 1} + \underbrace{D(\theta)}_{n_Y\times n_\varepsilon}\underbrace{\varepsilon_{t+1}}_{n_\varepsilon\times 1}, \end{aligned} \tag{3.4}$$

where $C(\theta) = \widehat{C}(\theta)A(\theta)$ and $D(\theta) = \widehat{C}(\theta)B(\theta)$.

3.2 Estimated Parameters

The estimated parameters are those governing the stochastic process

$$s_{t+1} = P_0 + Ps_t + Q\varepsilon_{s,t+1} \tag{3.5}$$

underlying the wedges and are thus the ones appearing in the matrices P_0, P and Q. More specifically, for the standard BCA model the stochastic process of the wedge shocks takes the form:

$$\begin{aligned} \underbrace{\begin{pmatrix} \log z_{t+1} \\ \tau_{lt+1} \\ \tau_{xt+1} \\ \log \hat{g}_{t+1} \end{pmatrix}}_{s_{t+1}} = \underbrace{\begin{pmatrix} \bar{z} \\ \bar{\tau}_l \\ \bar{\tau}_x \\ \bar{g} \end{pmatrix}}_{P_0} + \underbrace{\begin{pmatrix} \rho_z & \rho_{z,\tau_l} & \rho_{z,\tau_x} & \rho_{z,g} \\ \rho_{\tau_l,z} & \rho_{\tau_l} & \rho_{\tau_l,\tau_x} & \rho_{\tau_l,g} \\ \rho_{\tau_x,z} & \rho_{\tau_x,\tau_l} & \rho_{\tau_x} & \rho_{\tau_x,g} \\ \rho_{g,z} & \rho_{g,\tau_l} & \rho_{g,\tau_x} & \rho_g \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} \log z_t \\ \tau_{lt} \\ \tau_{xt} \\ \log \hat{g}_t \end{pmatrix}}_{s_t} \\ + \underbrace{\begin{pmatrix} q_{11} & 0 & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 \\ q_{31} & q_{32} & q_{33} & 0 \\ q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \varepsilon_{z,t+1} \\ \varepsilon_{\tau_l,t+1} \\ \varepsilon_{\tau_x,t+1} \\ \varepsilon_{g,t+1} \end{pmatrix}}_{\varepsilon_{s,t+1}}, \end{aligned} \tag{3.6}$$

The estimated steady state vector of wedge shocks $[\log z_{t+1}, \tau_{lt+1}, \tau_{xt+1}, \log \hat{g}_{t+1}]'$ are given by $(I_{4\times 4} - P)^{-1}P_0 = [\log(z)\ \tau_l\ \tau_x\ \log(\hat{g})]'$ which we define as $[z_{ss}\ \tau_{l_{ss}}\ \tau_{x_{ss}}\ g_{ss}]'$ for convenience.

In the monetary BCA model the stochastic process of the wedge shocks takes the form:

$$\underbrace{\begin{pmatrix} \log z_{t+1} \\ \tau_{lt+1} \\ \tau_{xt+1} \\ \log \hat{g}_{t+1} \\ \tau_{bt+1} \\ \tilde{R}_{t+1} \end{pmatrix}}_{s_{t+1}} = \underbrace{\begin{pmatrix} \bar{z} \\ \bar{\tau}_l \\ \bar{\tau}_x \\ \bar{g} \\ \bar{\tau}_b \\ \bar{R} \end{pmatrix}}_{P_0} + \underbrace{\begin{pmatrix} \rho_z & \rho_{z,\tau_l} & \rho_{z,\tau_x} & \rho_{z,g} & \rho_{z,\tau_b} & \rho_{z,\tilde{R}} \\ \rho_{\tau_l,z} & \rho_{\tau_l} & \rho_{\tau_l,\tau_x} & \rho_{\tau_l,g} & \rho_{\tau_l,\tau_b} & \rho_{\tau_l,\tilde{R}} \\ \rho_{\tau_x,z} & \rho_{\tau_x,\tau_l} & \rho_{\tau_x} & \rho_{\tau_x,g} & \rho_{\tau_x,\tau_b} & \rho_{\tau_x,\tilde{R}} \\ \rho_{g,z} & \rho_{g,\tau_l} & \rho_{g,\tau_x} & \rho_g & \rho_{g,\tau_b} & \rho_{g,\tilde{R}} \\ \rho_{\tau_b,z} & \rho_{\tau_b,\tau_l} & \rho_{\tau_b,\tau_x} & \rho_{\tau_b,g} & \rho_{\tau_b} & \rho_{\tau_b,\tilde{R}} \\ \rho_{\tilde{R},z} & \rho_{\tilde{R},\tau_l} & \rho_{\tilde{R},\tau_x} & \rho_{\tilde{R},g} & \rho_{\tilde{R},\tau_b} & \rho_{\tilde{R}} \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} \log z_t \\ \tau_{lt} \\ \tau_{xt} \\ \log \hat{g}_t \\ \tau_{bt} \\ \tilde{R}_t \end{pmatrix}}_{s_t}$$

$$+ \underbrace{\begin{pmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 & 0 & 0 \\ q_{31} & q_{32} & q_{33} & 0 & 0 & 0 \\ q_{41} & q_{42} & q_{43} & q_{44} & 0 & 0 \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} & 0 \\ q_{61} & q_{62} & q_{63} & q_{64} & q_{65} & q_{66} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \varepsilon_{z,t+1} \\ \varepsilon_{\tau_l,t+1} \\ \varepsilon_{\tau_x,t+1} \\ \varepsilon_{g,t+1} \\ \varepsilon_{\tau_b,t+1} \\ \varepsilon_{\tilde{R},t+1} \end{pmatrix}}_{\varepsilon_{s,t+1}}. \tag{3.7}$$

The estimated steady state vector of wedge shocks $[\log z_{t+1}, \tau_{lt+1}, \tau_{xt+1}, \log \hat{g}_{t+1}, \tau_{bt+1}, \tilde{R}_{t+1}]$ is given by $(I_{6\times 6} - P)^{-1} P_0 = [\log(z)\ \tau_l\ \tau_x\ \log(\hat{g})\ \tau_b\ \tilde{R}]$ which, again for convenience, we define as $[z_{ss}\ \tau_{l_{ss}}\ \tau_{x_{ss}}\ g_{ss}\ \tau_{b_{ss}}\ \tilde{R}_{ss}]'$. Šustek (2011) imposes $\tau_{b_{ss}} = 0$ and $\tilde{R}_{ss} = 0$ in estimation. In the identification tests we explore, inter alia, the case where these two parameters are estimated as well.

In Section 4 we present results also for the case where the structural parameters of the respective models are included in the identification analysis.

3.3 Komunjer and Ng (2011) Test for Strict Identification

In a DSGE model two parameter vectors θ_0 and θ_1 are observationally equivalent if the spectral density matrix evaluated at θ_0 is equal to the spectral density matrix evaluated at θ_1. A parameter set $\theta_0 \in \Theta$ is defined to be locally identifiable from the autocovariances of Y_t if there exists an open neighboorhood of θ_0 such that θ_0 and θ_1 being observationally equivalent necessarily implies $\theta_1 = \theta_0$.

Formally, Komunjer and Ng (2011) state that two triples $(\theta_0,\ I_{n_X},\ I_{n_\varepsilon})$ and $(\theta_1,\ T,\ U)$ are observationally equivalent if

$$A(\theta_1) = T\ A(\theta_0)\ T^{-1} \tag{3.8}$$
$$B(\theta_1) = T\ B(\theta_0)\ U \tag{3.9}$$
$$C(\theta_1) = C(\theta_0)\ T^{-1} \tag{3.10}$$
$$D(\theta_1) = D(\theta_0)\ U \tag{3.11}$$
$$\Sigma(\theta_1) = U\ \Sigma(\theta_0)\ U^{-1} \tag{3.12}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ and $\Sigma(\cdot)$ are the matrices of the state space model described in (3.4) with T and U being full rank matrices.

A necessary sufficient condition for identification is thus checking that the mapping

$$\delta^S(\theta, T, U) = \begin{pmatrix} \text{vec}(T\ A(\theta)\ T^{-1}) \\ \text{vec}(T\ B(\theta)\ U) \\ \text{vec}(C(\theta)\ T^{-1}) \\ \text{vec}(D(\theta)\ U), \\ \text{vec}(U\ \Sigma(\theta)\ U^{-1}) \end{pmatrix} \tag{3.13}$$

has full rank. The *rank* condition for local identification at θ_0 when the state space system is square (i.e., $n_\varepsilon = n_Y$) is thus given by

$$rank\left(\Delta^S(\theta_0)\right) = n_\theta + n_X^2 + n_\varepsilon^2, \tag{3.14}$$

where

$$\begin{aligned} \Delta^S(\theta_0) &\equiv \left(\Delta^S_\Lambda(\theta_0), \Delta^S_T(\theta_0), \Delta^S_U(\theta_0)\right) \\ &\equiv \left(\frac{\partial\delta(\theta_0, I_{n_X}, I_{n_\varepsilon})}{\partial\theta}, \frac{\partial\delta(\theta_0, I_{n_X}, I_{n_\varepsilon})}{\partial \text{vec}\ T}, \frac{\partial\delta(\theta_0, I_{n_X}, I_{n_\varepsilon})}{\partial \text{vec}\ U}\right). \end{aligned} \tag{3.15}$$

They also establish the following necessary *order* condition for identification

$$n_\theta + n_X^2 + n_\varepsilon^2 \leq n_\Lambda^S = (n_X + n_Y)(n_X + n_\varepsilon) + n_\varepsilon(n_\varepsilon + 1)/2. \tag{3.16}$$

It requires the number of equations defined by δ^S to be at least as large as the number of unknowns in those equations and can be rewritten as

$$n_\theta \leq n_Y n_X + n_\varepsilon(n_X + n_Y - n_\varepsilon) + \frac{n_\varepsilon(n_\varepsilon + 1)}{2} \equiv n_\delta. \tag{3.17}$$

Minimality and left-invertibility of the state space system are maintained assumptions of these conditions and we thus verify that they hold in our analysis. The first condition gets fulfilled by rewriting (3.4) in the particular form

$$\widetilde{X}_{t+1} = \begin{pmatrix} X_{1,t+1} \\ X_{2,t+1} \end{pmatrix} = \begin{pmatrix} \widetilde{A}_1(\theta) & 0 \\ \widetilde{A}_2(\theta) & 0 \end{pmatrix} \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} + \begin{pmatrix} \widetilde{B}_1(\theta) \\ \widetilde{B}_2(\theta) \end{pmatrix} \varepsilon_{t+1}, \tag{3.18}$$

$$Y_{t+1} = \left(\widetilde{C}_1(\theta)\ \widetilde{C}_2(\theta)\right) \begin{pmatrix} X_{1,t+1} \\ X_{2,t+1} \end{pmatrix}, \tag{3.19}$$

so that

$$X_{1,t+1} = \underbrace{\widetilde{A}_1(\theta)}_{A(\theta)} X_{1,t} + \underbrace{\widetilde{B}_1(\theta)}_{B(\theta)} \varepsilon_{t+1}, \tag{3.20}$$

$$\begin{aligned} Y_{t+1} = &\underbrace{\left(\widetilde{C}_1(\theta)\widetilde{A}_1(\theta) + \widetilde{C}_2(\theta)\widetilde{C}_2(\theta)\right)}_{C(\theta)} X_{1,t} \\ &+ \underbrace{\left(\widetilde{C}_1(\theta)\widetilde{B}_1(\theta) + \widetilde{C}_2(\theta)\widetilde{B}_2(\theta)\right)}_{D(\theta)} \varepsilon_{t+1}. \end{aligned} \tag{3.21}$$

Left-invertibility is ensured by the following assumption: *For every* $\theta \in \Theta$*,* $rank\ \mathcal{P}(z;\theta) = n_X + n_\varepsilon$ *in* $|z| > 1$, where $\mathcal{P}(z;\theta) \equiv \begin{pmatrix} zI_{n_X} - A(\theta) & B(\theta) \\ -C(\theta) & D(\theta) \end{pmatrix}$, $z \in \mathbb{C}$.

3.4 Iskrev (2015) Test for Strict and Weak Identification

In the following exposition, we closely follow Iskrev (2015).

3.4.1 Preliminaries

The log-likelihood function of a sample $\mathcal{Y}_T$ of data can be obtained using the sequence of one-step ahead prediction errors $e_{t|t-1} = Y_t - \hat{C}(\theta)\hat{X}_{t|t-1}$. The latter can be easily constructed using the one-step ahead forecasts of the state vector $\hat{X}_{t|t-1}$ returned by the Kalman filter. Assuming that the structural shocks are Gaussian implies that the conditional distribution of $e_{t|t-1}$ is Gaussian as well, with mean zero and covariance matrix $S_{t|t-1} = \hat{C}(\theta)P_{t|t-1}\hat{C}(\theta)'$, where $P_{t|t-1} = \mathbb{E}\left(X_t - \hat{X}_{t|t-1}\right)\left(X_t - \hat{X}_{t|t-1}\right)'$ can also be obtained from the Kalman filter recursions and is the covariance matrix of the one-step ahead forecasts conditional on information up to time $t-1$. The log-likelihood of the sample is then given by

$$l_T(\theta) = const. - \frac{1}{2}\sum_{t=1}^{T}\log\left(|S_{t|t-1}|\right) - \frac{1}{2}\sum_{t=1}^{T} e'_{t|t-1}S^{-1}_{t|t-1}e_{t|t-1}, \tag{3.22}$$

Under some regularity conditions the maximum likelihood estimator $\tilde{\theta}_T$ is consistent, asymptotically efficient and asymptotically normally distributed with

$$\sqrt{T}(\tilde{\theta}_T - \theta_0) \xrightarrow{d} \mathbb{N}(0, \mathcal{I}_0^{-1}), \tag{3.23}$$

where $\mathcal{I}_0$ is is the asymptotic Fisher information matrix evaluated at the true value of θ. More formally,

$$\mathcal{I}_0 \equiv \lim_{T\to\infty}\left(\frac{1}{T}\mathcal{I}_T\right), \tag{3.24}$$

where $\mathcal{I}_T$ is the finite sample Fisher information matrix given by

$$\mathcal{I}_T \equiv \mathbb{E}\left[\left\{\frac{\partial l_T(\theta)}{\partial\theta'}\right\}'\left\{\frac{\partial l_T(\theta)}{\partial\theta'}\right\}\right]. \tag{3.25}$$

3.4.2 General Principles of Identification Analysis

Suppose that inference about the parameters of the model collected in the vector θ is made using a sample with T observations of a random vector Y with a known probability density function $p(\mathcal{Y}_T;\theta)$, where $\mathcal{Y}^T = [Y_1', ..., Y_T']'$. The latter, when considered as a function of θ, contains all available sample information about θ associated with the observed data. It is then straightforward to see that a prerequisite for successful inference about θ is that its values imply distinct values of the density function $p(\mathcal{Y}_T;\theta)$. More formally, a point $\theta_0 \in \Theta$ is said to be identified if

$$\Pr\left(p(\mathcal{Y}_T;\theta) = p(\mathcal{Y}_T;\theta_0)\right) = 1 \;\Rightarrow\; \theta = \theta_0. \tag{3.26}$$

That is, if the density function yields the same value when evaluated at θ and at θ_0 this implies that θ is equal to θ_0. Is is possible to rewrite this condition in terms of the log-likelihood function $l_T(\theta) \equiv \log P(\mathcal{Y}_T;\theta)$:

$$\mathbb{E}_0 l_T(\theta_0) \geq \mathbb{E}_0 l_T(\theta),\ \forall\theta. \tag{3.27}$$

This follows from Jensen's inequality, see Rao (1971), and the logarithmic function being concave. It implies that the function $H(\theta_0, \theta) \equiv \mathbb{E}_0 \left(l_T(\theta) - l_T(\theta_0) \right)$ achieves a maximum at $\theta = \theta_0$, and that θ_0 is identified if and only if the maximum is unique. The conditions for local uniqueness of a maximum at θ_0 are that (i) $\frac{\partial H(\theta_0,\theta)}{\partial \theta}|_{\theta=\theta_0} = 0$ and (ii) $\frac{\partial^2 H(\theta_0,\theta)}{\partial \theta \partial \theta'}|_{\theta=\theta_0}$ is negative definite. If the maximum at θ_0 is locally unique then θ_0 is locally identified, i.e., there exists an open neighborhood of θ_0 where (3.26) holds $\forall \theta$[1]. One can show that, see Bowden (1973), the condition in (i) is always fulfilled and that the Hessian matrix in (ii) is equal to the negative of the Fisher information matrix. This leads to the following result by Rothenberg (1971): *Let θ_0 be a regular point[2] of the information matrix $\mathcal{I}_T(\theta)$. Then θ_0 is locally identifiable if and only if $\mathcal{I}_T(\theta_0)$ is non-singular.*

In general, non-singularity of the Fisher information matrix is both necessary and sufficient for local identification[3]. The information matrix is singular whenever the expected log-likelihood function is flat at θ_0. In this case, due to the lack of the variability induced by the parameters on the log-likelihood function, it is impossible to make inference about the parameters even with an infinite sample of data. There are two reasons why this might occur, following Iskrev (2015) we call them the "sensitivity" and "collinearity" factors. Either the parameters have no effect on the expected log-likelihood ("lack of sensitivity"), or different parameter values induce the same changes in the expected log-likelihood ("perfect collinearity"). It is thus useful to formalize ideas in order to investigate to which extent the two channels are at work. This can be done by using the fact that the information matrix is equal to the covariance matrix of the scores and can thus be expressed as

$$\mathcal{I}_T(\theta_0) = \Delta^{1/2} \mathcal{R}_T(\theta_0) \Delta^{1/2}, \tag{3.28}$$

where $\Delta = \text{diag}\left(\mathcal{I}_T(\theta_0) \right)$ is a diagonal matrix containing the variances of the elements of the score vector, and $\mathcal{R}_T(\theta_0)$ is the correlation matrix of the score vector. Thus a parameter $\theta_i \in \theta$ is locally unidentifiable if:

(I) "Lack of sensitivity": The expected log-likelihood is not affected by small changes in θ_i, i.e,

$$\Delta_i \equiv \mathbb{E} \left(\frac{\partial l_T(\theta_0)}{\partial \theta_i} \right)^2 = -\mathbb{E} \left(\frac{\partial^2 l_T(\theta_0)}{\partial \theta_i^2} \right) = 0 \tag{3.29}$$

(II) "Perfect collinearity": The effect of small changes in θ_i on the expected log-likelihood can be offset by varying other parameters, i.e.,

$$\varrho_i \equiv \sqrt{1 - 1/\mathcal{R}_T^{ii}} = 1, \tag{3.30}$$

where $\mathcal{R}_T^{ii}$ is the i-th diagonal element of the inverse of $\mathcal{R}_T$. As Iskrev (2015) puts its "[t]he intution about the meaning of ϱ_i comes from a well-known property of the correlation matrix Tucker et al. (1972), which implies that ϱ_i is the coefficient of multiple correlation between the partial derivative of the log-likelihood with respect to θ_i and the partial derivatives of the log-likelihood with respect to the other elements of θ".

Conditions (I) and (II) characterize the case in which the expected log-likelihood is completely flat and the parameters are thus not identifiable in a strict sense. The case of weak identification, on the other hand, arises when the expected log-likelihood features little curvature with respect to some parameters. We delve further into this issue next.

[1] Global identification would extend the uniqueness requirement to the whole parameter space.

[2] A point is regular if it belongs to an open neighborhood where the rank of the matrix does not vary.

[3] Please refer to Iskrev (2015) for further details.

3.4.3 Identification Strength

Local identifiability guarantees, in general, consistent estimation of θ. The precision with which θ is estimated is governed by the curvature of the expected log-likelihood function in the neighborhood of θ_0, of which the rank conditions above are not informative. Identification is weak whenever small changes in θ do not induce sufficiently large changes in $l_T(\theta)$ or, equivalently, when small changes in $l_T(\theta)$ are associated with large changes in θ. By weak we mean that the estimates are prone to be imprecisely estimated even in the presence of an infinitely large sample of data. The degree of "weakness" is thus related to the degree of precision. The latter is not an absolute but rather a relative concept which varies according to the application at hand.

We already saw that a central tool when investigating whether a parameter is locally identifiable or not is the Fisher information matrix. This is because, as shown in Rothenberg (1971), the latter is indicative of the degree of curvature of the expected log-likelihood function. To understand the next logical step in the analysis, namely the relationship between the curvature and the precision of the Maximum Likelihood (ML) estimator $\hat{\theta}_T$ it is useful to recall its asymptotic distribution described in (3.23). It is then straightforward to see that $\mathcal{I}_T^{-1}(\theta_0)/T$ is the sample counterpart of the covariance matrix of $\hat{\theta}_T$ and, analogously, ${\mathcal{I}_T^{ii}}^{-1}(\theta_0)/T$ is the sample counterpart of the variance of $\hat{\theta}_i$.

Asymptotic efficiency of ML estimation implies that $\hat{\theta}_T$ has the smallest asymptotic covariance matrix within the class of consistent estimators. This follows directly from the fact that, according to the Cramér-Rao theorem, the lower bound of the asymptotic covariance of any consistent estimator θ is given by the inverse of its asymptotic information matrix $\mathcal{I}_0$. Concurrently, the covariance matrix of any unbiased estimator is bounded below by the inverse of the sample information matrix $\mathcal{I}_T$ so that $b_i \equiv {\mathcal{I}_T^{ii}}^{-1}$ represents the lower bound on the variance of any unbiased estimator θ_i. To measure identification strength we can thus construct bounds on one-standard deviation intervals for the individual parameters.

As shown in Iskrev (2015), it is possible to relate the size of the bounds to the potential roots of identification deficiencies. Indeed, using the decomposition of $\mathcal{I}_T(\theta)$ in (3.25) and the properties of the correlation matrix one obtains

$$b_i = \frac{1}{\Delta_i(1-\rho_i)^2}. \tag{3.31}$$

When can one ascribe the identification problem to the "sensitivity" or "correlation" factor? In the first case, we know that the parameter does exert an irrelevant or weak effect on the likelihood, in which case $\Delta_i \approx 0$. In the second case, the effects induced on the likelihood by one parameter are compensated, and thus made redundant, by changes in other parameters, in which case $\rho_i \approx 1$. In both cases, the sources of weak identification lead to a large b_i and make inference about a parameter value challenging at best.

Notice that the sensitivity factor alone cannot guarantee successful parameter identification. Indeed, even if Δ_i is large, nearly perfect collinear effects of a parameter θ_i with respect to the other parameters θ_{-i} lead to values of ρ_i close to 1, in which case identification remains weak. This example illustrates well the difference between the information about θ_i contained in the likelihood when the other parameters are known, see Δ_i, and when they are not known, see b_i. This second source of information is smaller and the difference is increasing in the "correlation" factor, see ϱ_i.

4 Results

In this section we report results for the tests by Komunjer and Ng (2011) and Iskrev (2015). Results for the case where investment adjustment costs are introduced can be found in Appendix A. In the BCA and MBCA models strict and local identification is checked at the parameter set estimated (or fixed, as it is the case for the deep parameters) in Chari et al. (2007) and Šustek (2011) respectively.

4.1 Komunjer and Ng (2011)

We explore whether a parameter is identifiable in a strict sense. To answer this question in population we show results of the test by Komunjer and Ng (2011) first. Since this test requires computing the rank of the matrix $\Delta^S(\theta_0)$ in (3.15) to check whether the rank condition for strict identification holds we report results for different tolerance levels. Indeed, the rank of a matrix is equal to the number of its nonzero eigenvalues which are found by numerical routines using a cutoff to establish whether they are sufficiently small. Matlab, for instance, uses the tolerance $\text{Tol} = \max(\text{size}(M))\text{EPS}(||M||)$, where EPS is the float point precision of M. As pointed out by Komunjer and Ng (2011) this default tolerance does not take into account the fact that the matrix $\Delta^S(\theta_0)$ is often sparse and can thus be misleading. Results are thus reported for 11 tolerance values ranging from a maximum tolerance of 1e-2 to a minimum of 1e-11, along with the Matlab default one.

To isolate the parameters which are not strictly identifiable even with an infinite sample of data combinations of parameters which cause full rank failures in the $\Delta^S(\theta_0)$ are searched by inspecting its change in rank and its null space. This is first done at a high tolerance level of 1e-3 to flag the most difficult parameters to identify and then at the lowest tolerance level for which identification fails in order to find additional problematic parameters. Komunjer and Ng (2011) choose the highest tolerance level of 1e-3 "on the grounds that the numerical derivatives are computed using a step size of 1e-3". In Appendix D we report results for the case where a Matlab selected measure of the step size is being used when computing numerical derivatives. When this other measure is used the results suggest more identification power in both models.

In general, the lower the tolerance level the higher the rank of the matrix since more of the smallest eigenvalues are considered to be numerically different from zero. Then, according to the rank-nullity theorem - which states that the sum of the rank and the nullity of a matrix is equal to its number of columns - the null space will be smaller as well. This would lead to think that the set of parameters which are flagged as troublesome should be larger the lower the tolerance level. However, the way the set of problematic parameters is found is by identifying the number of columns of the orthonormal basis for the null space of the matrix Δ_T^S - obtained via singular value decomposition - whose (absolute) sum of elements is greater than the tolerance value (i.e., numerically larger than zero). This is because the vectors contained in a basis must be linearly independent and a null vector would always be linearly dependent with the other vectors. Thus, the lower the tolerance level, the more columns (and associated parameters) will fall within this set, the lower is the null space of the matrix Δ_T^S and, hence, the smaller is the set of parameters which are not strictly identifiable.

Finally, we perform conditional identification tests. More specifically, when the rank condition for strict identification fails, we check which parameters, when restricted, can enable identification of the remaining parameters.

4.1.1 Chari et al. (2007) BCA Model

Table 1 reveals that the model's parameters are strictly identifiable at Tol=1e-11 and at the Matlab default tolerance level. When inspecting the null space of $\Delta^S(\theta_0)$ no problematic parameters are found. This result might be explained by the fact that the lower is the tolerance level, the smaller is the set of problematic parameters, as explained above.

Table 1: Komunjer and Ng Test Results BCA Model

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	29	25	15	51	40	62	0
e-03	29	25	16	54	45	69	0
e-04	29	25	16	54	45	69	0
e-05	29	25	16	54	45	69	0
e-06	29	25	16	54	45	69	0
e-07	29	25	16	54	45	69	0
e-08	30	25	16	54	46	70	0
e-09	30	25	16	54	46	70	0
e-10	30	25	16	55	46	70	0
e-11	30	25	16	55	46	71	1
Default=2.756906e-12	30	25	16	55	46	71	1
Required	30	25	16	55	46	71	1

Summary: $n_\theta = 30, n_X = 5, n_\varepsilon = 4$.
Order Condition: $n_\theta = 30, n_\delta = 50$.

Table 2: Komunjer and Ng Test Results BCA Model (Deep Parameters Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	33	25	15	52	44	64	0
e-03	35	25	16	57	51	71	0
e-04	35	25	16	57	51	71	0
e-05	35	25	16	58	51	71	0
e-06	35	25	16	58	51	71	0
e-07	36	25	16	58	52	72	0
e-08	36	25	16	59	52	74	0
e-09	37	25	16	60	53	75	0
e-10	37	25	16	60	53	76	0
e-11	37	25	16	60	53	76	0
Default=5.513812e-12	37	25	16	61	53	76	0
Required	37	25	16	62	53	78	1

Summary: $n_\theta = 37, n_X = 5, n_\varepsilon = 4$.
Order Condition: $n_\theta = 37, n_\delta = 50$.
Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, g_{ss}, $\rho_{\tau_l,z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$, q_{22}, g_n, g_z, β, ψ, σ,
Problematic Parameters at Tol=5.513812e-12: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, q_{21}, q_{31}, q_{22}, q_{32}, q_{42}, q_{33}, q_{43}, q_{44}, g_n, g_z, β, δ, ψ, σ, α,

Moving to the case where also the deep parameters of the model are included in the identification test reveals a different picture, as reported in Table 2. Indeed, the extended parameter set does not pass the test at any tolerance value and several problematic parameters are found. At Tol=1e-3 the latter mainly concern some steady state values of the wedge shocks, off-diagonal elements of the matrix P and all deep parameters. Once the tolerance is further lowered to Tol=Default also several off-diagonal elements of the Q matrix are flagged as troublesome. At the default tolerance level, the model's parameters might be strictly identifiable if at least two restrictions are imposed. Indeed, all matrices are full rank with the exception of Λ^S which is short rank by two. We thus check whether fixing some parameters alleviates the strict identification deficiencies. As emerges from Table 3 there are several sets of restricted parameter combinations which allow to do so.

Table 3: Komunjer and Ng Conditional Test Results BCA Model, Tol = 5.513812e-12

Fixed	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
$\tau_{l_{ss}}$ z_{ss}	37	25	16	62	53	78	1
$\rho_{\tau_l,z}$ z_{ss}	37	25	16	62	53	78	1
ρ_{z,τ_l} z_{ss}	37	25	16	62	53	78	1
ρ_{τ_x,τ_l} z_{ss}	37	25	16	62	53	78	1
ρ_{g,τ_l} z_{ss}	37	25	16	62	53	78	1
ρ_{τ_l,τ_x} z_{ss}	37	25	16	62	53	78	1
$\rho_{\tau_l,g}$ z_{ss}	37	25	16	62	53	78	1
q_{21} z_{ss}	37	25	16	62	53	78	1
q_{22} z_{ss}	37	25	16	62	53	78	1
ψ z_{ss}	37	25	16	62	53	78	1
g_{ss} $\tau_{l_{ss}}$	37	25	16	62	53	78	1
g_n $\tau_{l_{ss}}$	37	25	16	62	53	78	1
g_z $\tau_{l_{ss}}$	37	25	16	62	53	78	1
β $\tau_{l_{ss}}$	37	25	16	62	53	78	1
$\rho_{\tau_l,z}$ g_{ss}	37	25	16	62	53	78	1
ρ_{z,τ_l} g_{ss}	37	25	16	62	53	78	1
ρ_{τ_x,τ_l} g_{ss}	37	25	16	62	53	78	1
ρ_{g,τ_l} g_{ss}	37	25	16	62	53	78	1
ρ_{τ_l,τ_x} g_{ss}	37	25	16	62	53	78	1
$\rho_{\tau_l,g}$ g_{ss}	37	25	16	62	53	78	1
q_{21} g_{ss}	37	25	16	62	53	78	1
q_{22} g_{ss}	37	25	16	62	53	78	1
ψ g_{ss}	37	25	16	62	53	78	1
g_n $\rho_{\tau_l,z}$	37	25	16	62	53	78	1
g_z $\rho_{\tau_l,z}$	37	25	16	62	53	78	1
β $\rho_{\tau_l,z}$	37	25	16	62	53	78	1
g_n ρ_{z,τ_l}	37	25	16	62	53	78	1
g_z ρ_{z,τ_l}	37	25	16	62	53	78	1
β ρ_{z,τ_l}	37	25	16	62	53	78	1
g_n ρ_{τ_x,τ_l}	37	25	16	62	53	78	1
g_z ρ_{τ_x,τ_l}	37	25	16	62	53	78	1
β ρ_{τ_x,τ_l}	37	25	16	62	53	78	1
g_n ρ_{g,τ_l}	37	25	16	62	53	78	1
g_z ρ_{g,τ_l}	37	25	16	62	53	78	1
β ρ_{g,τ_l}	37	25	16	62	53	78	1
$\rho_{\tau_l,g}$ ρ_{τ_l,τ_x}	37	25	16	62	53	78	1
g_n ρ_{τ_l,τ_x}	37	25	16	62	53	78	1
g_z ρ_{τ_l,τ_x}	37	25	16	62	53	78	1
β ρ_{τ_l,τ_x}	37	25	16	62	53	78	1
g_n $\rho_{\tau_l,g}$	37	25	16	62	53	78	1
g_z $\rho_{\tau_l,g}$	37	25	16	62	53	78	1
β $\rho_{\tau_l,g}$	37	25	16	62	53	78	1
g_n q_{21}	37	25	16	62	53	78	1
g_z q_{21}	37	25	16	62	53	78	1
β q_{21}	37	25	16	62	53	78	1
g_n q_{22}	37	25	16	62	53	78	1
g_z q_{22}	37	25	16	62	53	78	1
β q_{22}	37	25	16	62	53	78	1
ψ g_n	37	25	16	62	53	78	1
ψ g_z	37	25	16	62	53	78	1
ψ β	37	25	16	62	53	78	1
Required	37	25	16	62	53	78	1

Summary: $n_\theta = 37, n_X = 5, n_\varepsilon = 4$.

Order Condition: $n_\theta = 37, n_\delta = 50$.

4.1.2 Šustek (2011) Monetary BCA Model

The results for the monetary BCA model by Šustek (2011) resemble the ones just reported for the standard BCA model. Indeed the monetary BCA model fulfills the rank condition for strict identification established by Komunjer and Ng (2011) at Tol=1e-11 and at the Matlab default tolerance when the baseline set of estimated parameters is considered (Table 4) or when the latter contains also $[\tau_{b_{ss}}, \tilde{R}_{ss}]$ (Table 5). This is no longer true once the deep parameters are included in estimation. Indeed, also in this case, several steady state wedge shocks and off-diagonal elements of the P and Q matrix are not strictly identifiable, though this is only true when both $[\tau_{b_{ss}}, \tilde{R}_{ss}]$ and the deep parameters of the model are included in estimation (Table 8). Indeed, when only the deep parameters of the model are included in estimation on top of the baseline set of parameters considered in Šustek (2011) then this is only true at a low Tol=Default since at Tol=1e-3 only a few steady state wedge shocks are found to not meet the requirements for strict identifiability (Table 6). A similar pattern emerges once investment adjustment costs are introduced in the model (see Appendix A).

As to the conditional identification tests, we find that when the parameter set also includes the deep parameters of the model it is possible to fix some model parameters so as to make the rest identifiable (see Table 7) only when $\tau_{b_s s}$ and $\tilde{R}_{SS}$ are not included in estimation. The sets which restrict the least number of parameters are two and are found at Tol=1e-11, namely (i) $\{\pi_{SS},\ z_{SS}\}$ and (ii) $\{\pi_{SS},\ g_{SS}\}$

Table 4: Komunjer and Ng Test Results MBCA Model

Tol	Δ_Λ^S	Δ_T^S	Δ_U^S	$\Delta_{\Lambda T}^S$	$\Delta_{\Lambda U}^S$	Δ^S	Pass
e-02	60	49	33	101	89	130	0
e-03	60	49	36	108	96	143	0
e-04	60	49	36	109	96	144	0
e-05	60	49	36	109	96	144	0
e-06	60	49	36	109	96	144	0
e-07	60	49	36	109	96	145	0
e-08	60	49	36	109	96	145	0
e-09	61	49	36	109	97	145	0
e-10	61	49	36	109	97	145	0
Default=1.165290e-11	61	49	36	110	97	146	1
e-11	61	49	36	110	97	146	1
Required	61	49	36	110	97	146	1

Summary: $n_\theta = 61, n_X = 7, n_\varepsilon = 6$.

Order Condition: $n_\theta = 61, n_\delta = 105$.

Table 5: Komunjer and Ng Test Results MBCA Model ($\tau_{b_{ss}}$ and $\tilde{R}_{ss}$ Estimated)

Tol	Δ_Λ^S	Δ_T^S	Δ_U^S	$\Delta_{\Lambda T}^S$	$\Delta_{\Lambda U}^S$	Δ^S	Pass
e-02	62	49	33	103	91	131	0
e-03	62	49	36	110	98	144	0
e-04	62	49	36	111	98	146	0
e-05	62	49	36	111	98	146	0
e-06	62	49	36	111	98	146	0
e-07	62	49	36	111	98	147	0
Default=1.193257e-08	62	49	36	111	98	147	0
e-08	62	49	36	111	98	147	0
e-09	63	49	36	111	99	147	0
e-10	63	49	36	112	99	147	0
e-11	63	49	36	112	99	148	1
Required	63	49	36	112	99	148	1

Summary: $n_\theta = 63, n_X = 7, n_\varepsilon = 6$.

Order Condition: $n_\theta = 63, n_\delta = 105$.

Problematic Parameters at Tol=1e-3: z_{ss}, g_{ss},

Problematic Parameters at Tol=1.000000e-10: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{21}, q_{31}, q_{51}, q_{22}, q_{32}, q_{42}, q_{52}, q_{33}, q_{53}, q_{63}, q_{44},

Table 6: Komunjer and Ng Test Results MBCA Model (Deep Parameters Estimated)

Tol	Δ_Λ^S	Δ_T^S	Δ_U^S	$\Delta_{\Lambda T}^S$	$\Delta_{\Lambda U}^S$	Δ^S	Pass
e-02	68	49	35	104	98	133	0
e-03	69	49	36	113	105	146	0
e-04	69	49	36	116	105	148	0
e-05	69	49	36	116	105	148	0
e-06	69	49	36	116	105	148	0
e-07	70	49	36	117	106	149	0
e-08	70	49	36	118	106	152	0
e-09	71	49	36	118	107	153	0
e-10	71	49	36	119	107	154	0
Default=2.330580e-11	72	49	36	120	108	155	0
e-11	72	49	36	120	108	155	0
Required	72	49	36	121	108	157	1

Summary: $n_\theta = 72, n_X = 7, n_\varepsilon = 6$.

Order Condition: $n_\theta = 72, n_\delta = 105$.

Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{22}, q_{53}, g_n, g_z, β, ψ, σ, ρ_R, ω_π, ω_y, π_{ss},

Problematic Parameters at Tol=1.000000e-11: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{11}, q_{21}, q_{31}, q_{51}, q_{61}, q_{22}, q_{32}, q_{42}, q_{52}, q_{62}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, q_{65}, q_{66}, g_n, g_z, β, δ, ψ, σ, α, ρ_R, ω_π, ω_y, π_{ss},

Table 7: Komunjer and Ng Conditional Test Results MBCA Model, Tol = 2.330580e-11

Fixed	Δ_Λ^S	Δ_T^S	Δ_U^S	$\Delta_{\Lambda T}^S$	$\Delta_{\Lambda U}^S$	Δ^S	Pass
π_{ss} z_{ss}	72	49	36	121	108	157	1
π_{ss} g_{ss}	72	49	36	121	108	157	1
Required	72	49	36	121	108	157	1

Summary: $n_\theta = 72, n_X = 7, n_\varepsilon = 6.$

Order Condition: $n_\theta = 72, n_\delta = 105.$

Table 8: Komunjer and Ng Test Results MBCA Model ($\tau_{b_{ss}}$, $\tilde{R}_{ss}$ and Deep Parameters Estimated)

Tol	Δ_Λ^S	Δ_T^S	Δ_U^S	$\Delta_{\Lambda T}^S$	$\Delta_{\Lambda U}^S$	Δ^S	Pass
e-02	70	49	35	106	100	134	0
e-03	71	49	36	114	107	147	0
e-04	71	49	36	118	107	149	0
e-05	71	49	36	118	107	150	0
e-06	71	49	36	118	107	150	0
e-07	72	49	36	119	108	151	0
Default=2.386514e-08	72	49	36	120	108	153	0
e-08	72	49	36	120	108	154	0
e-09	73	49	36	120	109	154	0
e-10	73	49	36	121	109	156	0
e-11	74	49	36	122	110	157	0
Required	74	49	36	123	110	159	1

Summary: $n_\theta = 74, n_X = 7, n_\varepsilon = 6.$

Order Condition: $n_\theta = 74, n_\delta = 105.$

Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{21}, q_{31}, q_{51}, q_{22}, q_{32}, q_{42}, q_{52}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, g_n, g_z, β, δ, ψ, σ, ρ_R, ω_π, ω_y, π_{ss},

Problematic Parameters at Tol=1.000000e-11: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, $\tau_{b_{ss}}$, $\tilde{R}_{ss}$, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{11}, q_{21}, q_{31}, q_{41}, q_{51}, q_{61}, q_{22}, q_{32}, q_{42}, q_{52}, q_{62}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, q_{65}, q_{66}, g_n, g_z, β, δ, ψ, σ, α, ρ_R, ω_π, ω_y, π_{ss},

4.2 Iskrev (2015)

In this subsection we show the results of the parameter identification analysis of the baseline BCA and MBCA models. We only summarize the results on the strict identification tests and then focus on the weak identification analysis.

We start with the standard BCA model in the case where the deep parameters of the model are estimated on top of the ones governing the VAR(1) process of the innovations to the wedges, for a total of 37 parameters . The rank of the information matrix is 33. Using population data it is possible to obtain g_n and g_z can assumed to be known since in our detrending it set to a value such that average detrended output is equal to zero[4]. With 35 parameters, the rank of the information matrix is still 33. By brute force, we check the rank of the information matrix for all possible combinations of 33 parameters out of the 35 (595 such combinations) being restricted and find that (i) 21 combinations deliver a rank of 31, (ii) 355 combinations give rank of 32 and (iii) 219 combinations result in a rank of 33. Thus, we can fix 219 possible pairs of parameters and identify the ones left unrestricted in the respective cases.

As to the MBCA model, we start with 74 parameters which corresponds to the case where the deep parameter of the model as well as the steady state innovations to the asset market and monetary policy wedge (τ_{bss} and $\tilde{R}_{ss}$) are estimated on top of the ones governing the VAR(1) process. Fixing g_n and g_z as discussed above leaves 72 parameters and the rank of the information matrix is 68. There are 1028790 possible combinations of 68 out of 72 parameters which can be fixed, for 16652 of them the resulting rank of the information matrix is 68. When all deep parameters are excluded from the analysis and thus only the wedge parameters are considered 63 parameters are left. In this case, the rank of the information matrix is 62 and fixing any one of the following set of parameters gives a full rank of 62: $\{\tau_{b_{ss}},\ \rho_{\tau_b,z},\ \rho_{\tau_b,\tau_l},\ \rho_{\tau_b,\tau_x},\ \rho_{\tau_b,g},\ \rho_{z,\tau_b},\ \rho_{\tau_l,\tau_b},\ \rho_{\tau_x,\tau_b},\ \rho_{g,\tau_b},\ \rho_{\tilde{R},\tau_b},\ \rho_{\tau_b,\tilde{R}},\ q_{51},\ q_{52},\ q_{53},\ q_{54},\ q_{55}\}$. For instance, fixing τ_{bss} as done in Šustek (2011) gives a full rank of 62. If the non-wedge parameters are added 71 parameters are estimated and the rank of the information matrix is 68. There are 57155 combinations of 68 out of 71 parameters and for 59 of them fixing the relevant parameters gives a rank of 68.

Next, we study whether the parameters are affected by weak identification problems. We report Cramér-Rao lower bounds (CRLBs) as a measure of parameter estimation uncertainty in absolute terms, and relative CRLBs, $rCRLB(\theta_i) = CRLB(\theta_i)/abs(\theta_i)$, as a measure of relative uncertainty. The CRLBs reflect the uncertainty which arises from both the "sensitivity" and the "collinearity" factors. This is because, as discussed in Section 3.4.2, the CRLB is the product of these two components. The first component concerns the sensitivity of the log-likelihood function with respect to θ_i whereas the second relates to the degree of collinearity between the derivative of the likelihood with respect to θ_i and the derivatives of the likelihood with respect to all other free parameters θ_{-i}.

In order to interpret them it is useful to recall the following facts. The sensitivity factor for a parameter reports the value of the conditional CRLB, i.e., the lower bound of uncertainty given that all other parameters are known and no collinearity is present. As to the collinearity factor, it is indicative of the increase in the CRLB when the other parameters are unknown as well. The magnitude of the sensitivity component, unlike the collinearity one, depends on the scale of the parameters and is thus divided by the respective parameter values.

[4]This detrending method is used in Brinca et al. (2016) and is found to make the maximum likelihood estimation procedure very robust across a large set of OECD countries.

4.2.1 Chari et al. (2007) BCA Model

Table 9 deals with the standard BCA model and the case where the structural parameters are assumed to be known. The first column shows the values of the parameters governing the stochastic process of the wedges, the second one the CRLBs and the third one the relative CRLBs. The latter measure can be used to compare the relative identification strength across parameters which take different values. For instance, z_{ss} is much worse identified than g_{ss}. This is because the CRLB of z_{ss} relative to its value is 6.3701 while only 0.3617 for g_{ss}. Further examination of Table 9 suggests that the worst identified parameters, using (arbitrarily) a cutoff value of 1, are z_{ss}, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{g,τ_x}, $\rho_{z,g}$, $\rho_{\tau_l,g}$, q_{21}, q_{31}, q_{41}, q_{32}, q_{42} and q_{44}.

Table 9: BCA, Parameter Identification (All Deep Parameters Fixed)

	value	*CRLB*	*rCRLB*
z_{ss}	-0.0239	0.1524	6.3701
$\tau_{l_{ss}}$	0.3279	0.2513	0.7662
$\tau_{x_{ss}}$	0.4834	0.3763	0.7783
g_{ss}	-1.5344	0.5550	0.3617
ρ_z	0.9800	0.0484	0.0494
$\rho_{\tau_l,z}$	-0.0330	0.0631	1.9123
$\rho_{\tau_x,z}$	-0.0702	0.1135	1.6161
$\rho_{g,z}$	0.0048	0.0627	13.0322
ρ_{z,τ_l}	-0.0138	0.0353	2.5588
ρ_{τ_l}	0.9564	0.0675	0.0706
ρ_{τ_x,τ_l}	-0.0460	0.1346	2.9252
ρ_{g,τ_l}	-0.0081	0.0578	7.1346
ρ_{z,τ_x}	-0.0117	0.0825	7.0314
ρ_{τ_l,τ_x}	-0.0451	0.0768	1.7024
ρ_{τ_x}	0.8962	0.0994	0.1109
ρ_{g,τ_x}	0.0488	0.0964	1.9744
$\rho_{z,g}$	0.0192	0.0791	4.1117
$\rho_{\tau_l,g}$	0.0569	0.0760	1.3357
$\rho_{\tau_x,g}$	0.1041	0.0866	0.8321
ρ_g	0.9711	0.0907	0.0934
q_{11}	0.0116	0.0007	0.0578
q_{21}	0.0014	0.0028	2.0187
q_{31}	-0.0105	0.0111	1.0555
q_{41}	-0.0006	0.0013	2.2905
q_{22}	0.0064	0.0004	0.0633
q_{32}	0.0010	0.0099	9.6266
q_{42}	0.0061	0.0085	1.3946
q_{33}	0.0158	0.0110	0.6912
q_{43}	0.0142	0.0034	0.2412
q_{44}	0.0046	0.0047	1.0269

Table 10 considers the case where all parameters in the BCA model are estimated with the exception of four out of the seven structural parameters in the BCA model being held fixed so as to guarantee strict identification of the remaining parameters. As described above, there are many such combinations which would have guaranteed a full rank information matrix. On top of leaving out g_n and g_z which we can calculate using additional data we leave out β and ψ from the analysis. Not surprisingly, the relative uncertainty measures increase. Moreover, the set of worst identified parameters includes also τ_{lss}, τ_{xss}, $\rho_{\tau_x,g}$ and q_{33}.

Table 10: BCA, Parameter Identification (Some Deep Parameters Free)

	$value$	$CRLB$	$rCRLB$
z_{ss}	-0.0239	1.7607	73.6039
$\tau_{l_{ss}}$	0.3279	0.4714	1.4376
$\tau_{x_{ss}}$	0.4834	1.5796	3.2674
g_{ss}	-1.5344	0.7628	0.4972
ρ_z	0.9800	0.0609	0.0622
$\rho_{\tau_l,z}$	-0.0330	0.0683	2.0712
$\rho_{\tau_x,z}$	-0.0702	0.1159	1.6503
$\rho_{g,z}$	0.0048	0.0695	14.4480
ρ_{z,τ_l}	-0.0138	0.0571	4.1391
ρ_{τ_l}	0.9564	0.0750	0.0784
ρ_{τ_x,τ_l}	-0.0460	0.1441	3.1325
ρ_{g,τ_l}	-0.0081	0.0830	10.2360
ρ_{z,τ_x}	-0.0117	0.1214	10.3566
ρ_{τ_l,τ_x}	-0.0451	0.0798	1.7692
ρ_{τ_x}	0.8962	0.1288	0.1437
ρ_{g,τ_x}	0.0488	0.0970	1.9863
$\rho_{z,g}$	0.0192	0.1046	5.4386
$\rho_{\tau_l,g}$	0.0569	0.1020	1.7917
$\rho_{\tau_x,g}$	0.1041	0.1243	1.1947
ρ_g	0.9711	0.1141	0.1175
q_{11}	0.0116	0.0057	0.4902
q_{21}	0.0014	0.0040	2.8614
q_{31}	-0.0105	0.0149	1.4217
q_{41}	-0.0006	0.0013	2.3349
q_{22}	0.0064	0.0035	0.5488
q_{32}	0.0010	0.0123	11.8844
q_{42}	0.0061	0.0117	1.9119
q_{33}	0.0158	0.0215	1.3562
q_{43}	0.0142	0.0046	0.3233
q_{44}	0.0046	0.0068	1.4834
δ	0.0118	0.0014	0.1215
σ	1.0000	0.5176	0.5176
α	0.3500	0.3171	0.9059

The "sensitivity" and "collinearity" components of the parameters' CRLBs for the case where the deep parameters are fixed are reported in Table 11 and labeled as "sens." and "coll.". It is immediate to see that the channel which drives weak identification is the collinearity one. Indeed, while all parameters exert a strong effect on the likelihood, some of them induce a variation in the likelihood which is very similar to other parameters. For some parameters the sensitivity factor is so strong that it outweighs the negative effect of the collinearity factor on their overall uncertainty. For others, however, the collinearity factor predominates and leads their relative uncertainty to be high. It is also interesting to take a look at the largest multiple correlation coefficients between $\partial l_T(\theta)/\partial\theta_i$ and $\partial l_T(\theta)/\partial\theta_{-i}$, namely $\varrho_{i(n)}$. We report the single and pairwise correlation coefficients in the columns labeled by $\varrho_{i(1)}$ and $\varrho_{i(2)}$ respectively. The problematic parameters exert an effect on the likelihood which is mostly collinear with the elements of the matrices they belong to in the VAR(1) process of the innovations to the wedges (i.e., P_0, P and Q). The fact that the steady state innovations to the wedges are strongly correlated with elements of both the P_0 and P matrix is due to the fact that they are obtained as $(I-P)^{-1}P_0$.

Table 11: BCA, Information Matrix Decomposition (Observing 4 Standard Variables)

	CRLB/para.	sens/para.	coll.	ϱ_i	$\varrho_{i(1)}$	$\varrho_{i(2)}$
z_{ss}	6.370	0.575	11.082	0.995920	0.886 (g_{ss})	0.931 (g_{ss}, $\rho_{g,z}$)
$\tau_{l_{ss}}$	0.766	0.018	41.833	0.999714	0.739 ($\tau_{x_{ss}}$)	0.905 ($\tau_{x_{ss}}$, ρ_{z,τ_x})
$\tau_{x_{ss}}$	0.778	0.009	82.492	0.999927	0.886 (g_{ss})	0.934 ($\tau_{l_{ss}}$, g_{ss})
g_{ss}	0.362	0.006	59.573	0.999859	0.886 ($\tau_{x_{ss}}$)	0.971 (z_{ss}, $\tau_{x_{ss}}$)
ρ_z	0.049	0.000	153.529	0.999979	0.988 ($\rho_{g,z}$)	0.991 ($\rho_{\tau_l,z}$, $\rho_{g,z}$)
$\rho_{\tau_l,z}$	1.912	0.030	64.792	0.999881	0.907 (ρ_z)	0.982 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
$\rho_{\tau_x,z}$	1.616	0.026	62.859	0.999873	0.908 ($\rho_{g,z}$)	0.974 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
$\rho_{g,z}$	13.032	0.097	134.773	0.999972	0.988 (ρ_z)	0.990 (ρ_z, $\rho_{\tau_x,z}$)
ρ_{z,τ_l}	2.559	0.014	176.574	0.999984	0.988 (ρ_{g,τ_l})	0.993 (ρ_{g,τ_l}, $\rho_{z,g}$)
ρ_{τ_l}	0.071	0.001	108.028	0.999957	0.978 ($\rho_{\tau_l,g}$)	0.990 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
ρ_{τ_x,τ_l}	2.925	0.026	112.667	0.999961	0.969 ($\rho_{\tau_x,g}$)	0.986 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
ρ_{g,τ_l}	7.135	0.036	196.423	0.999987	0.988 (ρ_{z,τ_l})	0.993 (ρ_{g,τ_x}, ρ_g)
ρ_{z,τ_x}	7.031	0.012	600.752	0.999999	0.988 (ρ_{g,τ_x})	0.999 (ρ_z, $\rho_{z,g}$)
ρ_{τ_l,τ_x}	1.702	0.009	181.791	0.999985	0.983 ($\rho_{\tau_l,g}$)	0.998 ($\rho_{\tau_l,z}$, $\rho_{\tau_l,g}$)
ρ_{τ_x}	0.111	0.001	126.383	0.999969	0.978 ($\rho_{\tau_x,g}$)	0.997 ($\rho_{\tau_x,z}$, $\rho_{\tau_x,g}$)
ρ_{g,τ_x}	1.974	0.004	474.827	0.999998	0.988 (ρ_{z,τ_x})	0.999 ($\rho_{g,z}$, ρ_g)
$\rho_{z,g}$	4.112	0.005	881.973	0.999999	0.988 (ρ_g)	0.999 (ρ_z, ρ_{z,τ_x})
$\rho_{\tau_l,g}$	1.336	0.005	272.329	0.999993	0.983 (ρ_{τ_l,τ_x})	0.999 ($\rho_{\tau_l,z}$, ρ_{τ_l,τ_x})
$\rho_{\tau_x,g}$	0.832	0.005	163.595	0.999981	0.978 (ρ_{τ_x})	0.998 ($\rho_{\tau_x,z}$, ρ_{τ_x})
ρ_g	0.093	0.000	684.987	0.999999	0.988 ($\rho_{z,g}$)	0.999 ($\rho_{g,z}$, ρ_{g,τ_x})
q_{11}	0.058	0.036	1.628	0.788970	0.780 (q_{31})	0.788 (q_{21}, q_{31})
q_{21}	2.019	0.247	8.160	0.992462	0.747 (q_{41})	0.756 (q_{11}, q_{41})
q_{31}	1.056	0.038	27.708	0.999349	0.951 (q_{41})	0.960 (q_{11}, q_{41})
q_{41}	2.291	0.654	3.504	0.958421	0.951 (q_{31})	0.958 (q_{21}, q_{31})
q_{22}	0.063	0.045	1.399	0.699484	0.622 (q_{42})	0.632 ($\tau_{l_{ss}}$, q_{42})
q_{32}	9.627	0.388	24.829	0.999189	0.951 (q_{42})	0.951 ($\tau_{l_{ss}}$, q_{42})
q_{42}	1.395	0.062	22.658	0.999026	0.951 (q_{32})	0.955 (q_{22}, q_{32})
q_{33}	0.691	0.024	28.643	0.999390	0.909 (q_{43})	0.911 (ρ_{τ_x}, q_{43})
q_{43}	0.241	0.027	9.087	0.993927	0.909 (q_{33})	0.909 (z_{ss}, q_{33})
q_{44}	1.027	0.058	17.694	0.998402	0.088 ($\tau_{l_{ss}}$)	0.173 ($\tau_{l_{ss}}$, $\tau_{x_{ss}}$)

Table 12 presents the decomposition of parameter uncertainty into a sensitivity and collinearity component for the case where also some deep parameters are estimated. The conclusions drawn about identification patterns drawn above are largely unaffected. The only key difference is that steady state parameters are mainly collinear with the deep parameters of the model and viceversa. This is due to the fact that the latter are directly informative about each other via the steady state equations of the model.

Table 12: BCA (Some Deep Parameters Estimated), Information Matrix Decomposition

	CRLB/para.	sens/para.	coll.	ϱ_i	$\varrho_{i(1)}$	$\varrho_{i(2)}$
z_{ss}	73.604	0.575	128.045	0.999970	0.886 (g_{ss})	0.955 (g_{ss}, α)
$\tau_{l_{ss}}$	1.438	0.018	78.490	0.999919	0.887 (α)	0.909 (z_{ss}, α)
$\tau_{x_{ss}}$	3.267	0.009	346.308	0.999996	0.900 (δ)	0.970 (ρ_g, α)
g_{ss}	0.497	0.006	81.884	0.999925	0.886 ($\tau_{x_{ss}}$)	0.971 (z_{ss}, $\tau_{x_{ss}}$)
ρ_z	0.062	0.000	193.365	0.999987	0.988 ($\rho_{g,z}$)	0.991 ($\rho_{\tau_l,z}$, $\rho_{g,z}$)
$\rho_{\tau_l,z}$	2.071	0.030	70.178	0.999898	0.907 (ρ_z)	0.982 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
$\rho_{\tau_x,z}$	1.650	0.026	64.190	0.999879	0.908 ($\rho_{g,z}$)	0.974 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
$\rho_{g,z}$	14.448	0.097	149.415	0.999978	0.988 (ρ_z)	0.990 (ρ_z, $\rho_{\tau_x,z}$)
ρ_{z,τ_l}	4.139	0.014	285.622	0.999994	0.988 (ρ_{g,τ_l})	0.993 (ρ_{g,τ_l}, $\rho_{z,g}$)
ρ_{τ_l}	0.078	0.001	120.051	0.999965	0.978 ($\rho_{\tau_l,g}$)	0.990 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
ρ_{τ_x,τ_l}	3.132	0.026	120.650	0.999966	0.969 ($\rho_{\tau_x,g}$)	0.986 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
ρ_{g,τ_l}	10.236	0.036	281.808	0.999994	0.988 (ρ_{z,τ_l})	0.993 (ρ_{g,τ_x}, ρ_g)
ρ_{z,τ_x}	10.357	0.012	884.848	0.999999	0.988 (ρ_{g,τ_x})	0.999 (ρ_z, $\rho_{z,g}$)
ρ_{τ_l,τ_x}	1.769	0.009	188.922	0.999986	0.983 ($\rho_{\tau_l,g}$)	0.998 ($\rho_{\tau_l,z}$, $\rho_{\tau_l,g}$)
ρ_{τ_x}	0.144	0.001	163.762	0.999981	0.978 ($\rho_{\tau_x,g}$)	0.997 ($\rho_{\tau_x,z}$, $\rho_{\tau_x,g}$)
ρ_{g,τ_x}	1.986	0.004	477.688	0.999998	0.988 (ρ_{z,τ_x})	0.999 ($\rho_{g,z}$, ρ_g)
$\rho_{z,g}$	5.439	0.005	1166.579	1.000000	0.988 (ρ_g)	0.999 (ρ_z, ρ_{z,τ_x})
$\rho_{\tau_l,g}$	1.792	0.005	365.308	0.999996	0.983 (ρ_{τ_l,τ_x})	0.999 ($\rho_{\tau_l,z}$, ρ_{τ_l,τ_x})
$\rho_{\tau_x,g}$	1.195	0.005	234.875	0.999991	0.978 (ρ_{τ_x})	0.998 ($\rho_{\tau_x,z}$, ρ_{τ_x})
ρ_g	0.118	0.000	862.241	0.999999	0.988 ($\rho_{z,g}$)	0.999 ($\rho_{g,z}$, ρ_{g,τ_x})
q_{11}	0.490	0.036	13.799	0.997371	0.780 (q_{31})	0.788 (q_{21}, q_{31})
q_{21}	2.861	0.247	11.566	0.996255	0.747 (q_{41})	0.756 (q_{11}, q_{41})
q_{31}	1.422	0.038	37.321	0.999641	0.951 (q_{41})	0.960 (q_{11}, q_{41})
q_{41}	2.335	0.654	3.572	0.960019	0.951 (q_{31})	0.958 (q_{21}, q_{31})
q_{22}	0.549	0.045	12.138	0.996600	0.622 (q_{42})	0.632 ($\tau_{l_{ss}}$, q_{42})
q_{32}	11.884	0.388	30.653	0.999468	0.951 (q_{42})	0.951 ($\tau_{l_{ss}}$, q_{42})
q_{42}	1.912	0.062	31.063	0.999482	0.951 (q_{32})	0.955 (q_{22}, q_{32})
q_{33}	1.356	0.024	56.198	0.999842	0.909 (q_{43})	0.911 (ρ_{τ_x}, q_{43})
q_{43}	0.323	0.027	12.183	0.996626	0.909 (q_{33})	0.909 (z_{ss}, q_{33})
q_{44}	1.483	0.058	25.559	0.999234	0.408 (σ)	0.660 ($\tau_{x_{ss}}$, σ)
δ	0.122	0.022	5.444	0.982983	0.900 ($\tau_{x_{ss}}$)	0.949 (z_{ss}, α)
σ	0.518	0.015	35.581	0.999605	0.752 ($\tau_{x_{ss}}$)	0.868 ($\tau_{x_{ss}}$, q_{44})
α	0.906	0.003	292.762	0.999994	0.887 ($\tau_{l_{ss}}$)	0.973 ($\tau_{x_{ss}}$, ρ_g)

We thus conclude that weak identification can be attributed to the fact that parameters in the steady state vector, in the autoregressive matrix and in standard deviations of the fundamental innovations have a similar effect on the likelihood as other parameters appearing within the matrices they belong to. In order to understand how this pattern comes about we perform the following experiment. First, we assume that the observed variables are not output, investment, hours and government consumption (i.e., log $\hat{y}_t$, log $\hat{x}_t$, log l_t, log $\hat{g}_t$) but rather the four innovations to the wedges (i.e., $\log(z_t)$, τ_{lt}, τ_{xt}, $\log(\hat{g}_t)$). Second, we assume that the parameters governing the VAR(1) process of the innovations to the wedges are estimated in isolation of the BCA model. In other words, we perform identification analysis on the parameters assuming that a VAR is run directly on the wedges, which are treated as observable in this experiment. This allows us to shed light on the question whether the weak identification patterns described above are an intrinsic property of the model or of the VAR(1) process itself. Table 13 presents the information matrix decomposition for the case just described.

Table 13: BCA Model, Information Matrix Decomposition (Observing 4 Wedges)

	CRLB/para.	sens/para.	coll.	ϱ_i	$\varrho_{i(1)}$	$\varrho_{i(2)}$
z_{ss}	8.528	0.137	62.163	0.999871	0.997 (g_{ss})	0.998 ($\tau_{l_{ss}}$, g_{ss})
$\tau_{l_{ss}}$	1.062	0.013	79.708	0.999921	0.985 (g_{ss})	0.998 ($\tau_{x_{ss}}$, g_{ss})
$\tau_{x_{ss}}$	1.007	0.004	251.762	0.999992	0.999 (g_{ss})	1.000 ($\tau_{l_{ss}}$, g_{ss})
g_{ss}	0.501	0.001	390.297	0.999997	0.999 ($\tau_{x_{ss}}$)	1.000 ($\tau_{l_{ss}}$, $\tau_{x_{ss}}$)
ρ_z	0.053	0.002	30.412	0.999459	0.937 (ρ_{z,τ_l})	0.994 (ρ_{z,τ_x}, $\rho_{z,g}$)
$\rho_{\tau_l,z}$	0.893	0.042	21.480	0.998916	0.938 (ρ_{τ_l})	0.993 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
$\rho_{\tau_x,z}$	1.206	0.023	51.881	0.999814	0.935 (ρ_{τ_x,τ_l})	0.993 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
$\rho_{g,z}$	15.015	0.323	46.481	0.999769	0.934 (ρ_{g,τ_l})	0.993 (ρ_{g,τ_x}, ρ_g)
ρ_{z,τ_l}	2.922	0.074	39.262	0.999676	0.991 ($\rho_{z,g}$)	0.996 (ρ_{z,τ_x}, $\rho_{z,g}$)
ρ_{τ_l}	0.024	0.001	27.282	0.999328	0.990 ($\rho_{\tau_l,g}$)	0.996 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
ρ_{τ_x,τ_l}	1.433	0.022	65.457	0.999883	0.990 ($\rho_{\tau_x,g}$)	0.996 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
ρ_{g,τ_l}	6.950	0.118	58.772	0.999855	0.989 (ρ_g)	0.995 (ρ_{g,τ_x}, ρ_g)
ρ_{z,τ_x}	7.522	0.061	123.966	0.999967	0.992 ($\rho_{z,g}$)	0.999 (ρ_z, $\rho_{z,g}$)
ρ_{τ_l,τ_x}	1.110	0.013	84.669	0.999930	0.992 ($\rho_{\tau_l,g}$)	0.999 ($\rho_{\tau_l,z}$, $\rho_{\tau_l,g}$)
ρ_{τ_x}	0.161	0.001	204.424	0.999988	0.991 ($\rho_{\tau_x,g}$)	0.999 ($\rho_{\tau_x,z}$, $\rho_{\tau_x,g}$)
ρ_{g,τ_x}	2.512	0.014	182.688	0.999985	0.991 (ρ_g)	0.999 ($\rho_{g,z}$, ρ_g)
$\rho_{z,g}$	4.432	0.024	185.122	0.999985	0.992 (ρ_{z,τ_x})	1.000 (ρ_z, ρ_{z,τ_x})
$\rho_{\tau_l,g}$	0.851	0.007	127.227	0.999969	0.992 (ρ_{τ_l,τ_x})	1.000 ($\rho_{\tau_l,z}$, ρ_{τ_l,τ_x})
$\rho_{\tau_x,g}$	1.341	0.004	306.941	0.999995	0.991 (ρ_{τ_x})	1.000 ($\rho_{\tau_x,z}$, ρ_{τ_x})
ρ_g	0.122	0.000	274.011	0.999993	0.991 (ρ_{g,τ_x})	1.000 ($\rho_{g,z}$, ρ_{g,τ_x})
q_{11}	0.058	0.035	1.630	0.789720	0.781 (q_{31})	0.789 (q_{21}, q_{31})
q_{21}	0.378	0.247	1.530	0.756825	0.748 (q_{41})	0.757 (q_{11}, q_{41})
q_{31}	0.137	0.038	3.593	0.960488	0.951 (q_{41})	0.960 (q_{11}, q_{41})
q_{41}	2.292	0.652	3.515	0.958676	0.951 (q_{31})	0.959 (q_{21}, q_{31})
q_{22}	0.058	0.045	1.282	0.625594	0.624 (q_{42})	0.625 (q_{32}, q_{42})
q_{32}	1.258	0.387	3.253	0.951570	0.951 (q_{42})	0.952 (q_{22}, q_{42})
q_{42}	0.208	0.061	3.384	0.955345	0.951 (q_{32})	0.955 (q_{22}, q_{32})
q_{33}	0.058	0.024	2.404	0.909374	0.909 (q_{43})	0.909 (ρ_{τ_l,τ_x}, q_{43})
q_{43}	0.064	0.026	2.404	0.909367	0.909 (q_{33})	0.909 (ρ_{τ_l,τ_x}, q_{33})
q_{44}	0.058	0.058	1.000	0.008064	0.002 (q_{31})	0.003 (ρ_{g,τ_l}, ρ_g)

The main takeaways are that (i) the collinearity patterns are preserved and (ii) for almost all steady state innovations to the wedges, collinearity is higher. The intuition for result (ii) is that by fixing deep parameters the model imposes additional restrictions on the steady states of the innovations to the wedges via the steady state equations. This results in additional information and, therefore, improved identification strength.

To further develop intuition and visualize our findings, we plot pairwise correlations coefficients between $\partial l_T(\theta)/\partial\theta_i$ and $\partial l_T(\theta)/\partial\theta_{-i}$ across the entire parameter space in Figures 1 and 2. Clearly, correlation coefficients between the derivatives with respect to the same parameter are equal to 1, marked as dark red squares on the main diagonal. The following results emerge. First, the figures confirm the main finding that the steady state innovations to the wedges $(I-P)^{-1}P_0$, the elements of the autoregressive matrix P and of the Choleski decomposition of the variance covariance matrix Q induce variation in the likelihood which is mainly similar to that generated by other parameters in the matrix they belong to. Indeed, in Figure 2 the strongest collinearity is among elements of the same matrix and only some elements of the P and Q have a weakly collinear effect. Second, collinearity is less pervasive when innovations to the wedges are assumed to be observed and the parameters are estimated using the VAR in isolation from the model. The additional links between parameters and their likelihood introduced by the model equations are thus not sufficiently rich to let each parameter in the VAR(1) matrices P_0, P and Q raise its own, distinct voice. In other words, estimating the wedge parameters within a model does not obviate the collinearity patterns which emerge in isolation from it.

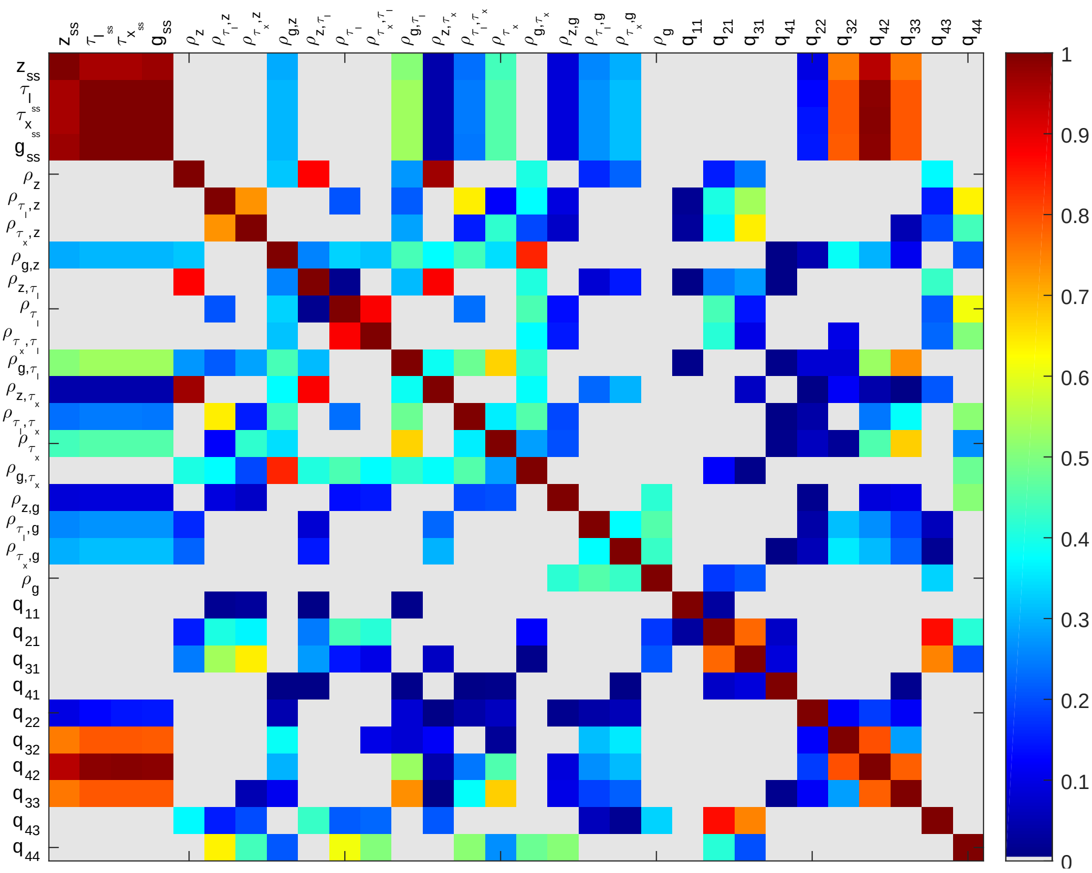

Figure 1: BCA, Pairwise Correlations (Observing 4 Standard Variables)

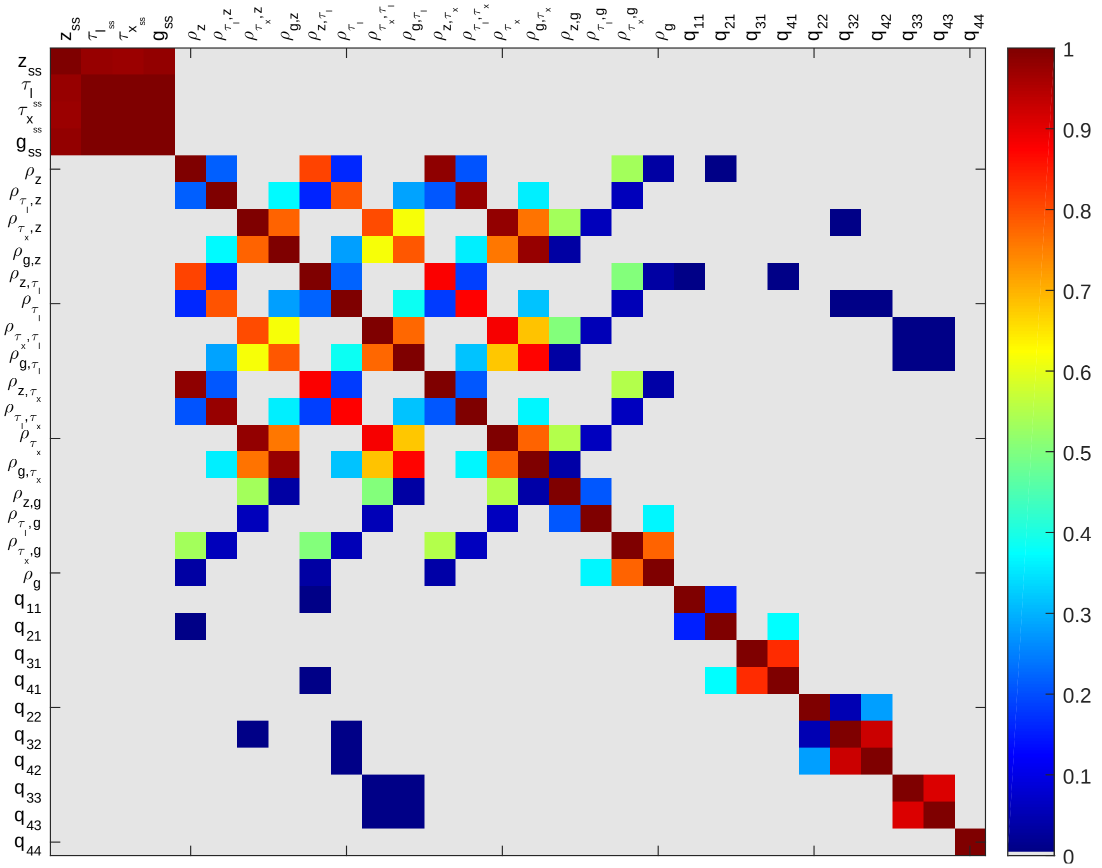

Figure 2: BCA, Pairwise Correlations (Observing 4 Wedges)

4.2.2 Šustek (2011) Monetary BCA Model

Tables 14 and 16 contain information about the relative CRLBs as well as about the sensitivity and collinearity components for the MBCA model wedge parameters. Using again a cutoff value of 1 the set of worst identified parameters is given by z_{ss}, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, q_{41}, q_{51}, q_{61}, q_{32}, q_{52}, q_{53}, q_{63}, q_{54}, q_{64} and q_{66}. This amounts to 39 out of 61 parameters, i.e., roughly two thirds of the estimated parameters are weakly identified. As in the standard BCA model these parameters concern the steady state innovation to the efficiency wedge z_{ss} as well as the off-diagonal elements of the P and Q matrices. Identification deficiencies can again be attributed to the strong collinearity between the effects that parameters within the same matrix exert on the likelihood. We focus on the case where some deep parameters are included in the identification analysis in Tables 15 and 17. Also in this case, we keep fixed the other deep parameters so as to guarantee a full rank information matrix. Among the many available choices we choose to leave out β, ψ as well as steady state inflation π_{ss} and the weight on inflation in the Taylor rule ω_π. Again, relative CRLBs increase as expected. Additional parameters which are weakly identified are τ_{lss}, τ_{xss}, $\rho_{z,\tilde{R}}$, q_{31}, q_{33}, q_{43}, δ, σ, α, ω_π. The collinearity patterns are similar to the ones in the BCA model.

Finally, we carry out the identification analysis on the wedge parameters with the twist of treating the innovations to the wedges as observables for the MBCA model as well. We again find that (i) parameters of the same matrices having a collinear effect on the likelihood is a property of the VAR process itself (Table 18) and (ii) collinearity is less pervasive when the parameters are estimated in isolation to the model (Figures 3 and 4).

Table 14: MBCA, Parameter Identification (All Deep Parameters, $\tau_{b_{ss}}$ and $\tilde{R}_{ss}$ fixed)

	value	*CRLB*	*rCRLB*
z_{ss}	-0.0246	0.1333	5.4109
$\tau_{l_{ss}}$	0.1267	0.0345	0.2720
$\tau_{x_{ss}}$	0.4649	0.1310	0.2818
g_{ss}	-1.5389	0.0592	0.0385
$\tilde{R}_{ss}$	0.0000	0.0023	0.0023
ρ_z	0.8541	0.2833	0.3317
$\rho_{\tau_l,z}$	-0.0437	0.1504	3.4424
$\rho_{\tau_x,z}$	-0.0557	0.1393	2.5014
$\rho_{g,z}$	0.0533	0.1774	3.3285
$\rho_{\tau_b,z}$	0.0633	0.4210	6.6544
$\rho_{\tilde{R},z}$	-0.0141	0.0229	1.6232
ρ_{z,τ_l}	-0.1482	0.4083	2.7553
ρ_{τ_l}	1.0580	0.2108	0.1993
ρ_{τ_x,τ_l}	-0.0335	0.1629	4.8637
ρ_{g,τ_l}	0.0587	0.2139	3.6444
ρ_{τ_b,τ_l}	-0.2979	0.6935	2.3279
$\rho_{\tilde{R},\tau_l}$	0.0146	0.0295	2.0178
ρ_{z,τ_x}	0.2654	0.3643	1.3724
ρ_{τ_l,τ_x}	-0.0014	0.2048	146.3072
ρ_{τ_x}	1.0877	0.1840	0.1692
ρ_{g,τ_x}	-0.0974	0.2229	2.2881
ρ_{τ_b,τ_x}	0.0850	0.6202	7.2964
$\rho_{\tilde{R},\tau_x}$	0.0005	0.0358	71.6947
$\rho_{z,g}$	-0.0094	0.1084	11.4917
$\rho_{\tau_l,g}$	0.0097	0.0619	6.3770
$\rho_{\tau_x,g}$	0.0026	0.0326	12.5353
ρ_g	1.0053	0.0479	0.0477
$\rho_{\tau_b,g}$	-0.0076	0.2253	29.6442
$\rho_{\tilde{R},g}$	0.0004	0.0088	21.8878
ρ_{z,τ_b}	-0.0654	0.1679	2.5674
ρ_{τ_l,τ_b}	0.0465	0.0865	1.8607
ρ_{τ_x,τ_b}	-0.0116	0.0763	6.5772
ρ_{g,τ_b}	0.0241	0.0940	3.9020
ρ_{τ_b}	0.8263	0.2712	0.3282
$\rho_{\tilde{R},\tau_b}$	0.0063	0.0137	2.1719
$\rho_{z,\tilde{R}}$	0.7994	0.7968	0.9967
$\rho_{\tau_l,\tilde{R}}$	-0.7219	0.4557	0.6313
$\rho_{\tau_x,\tilde{R}}$	0.4016	0.5299	1.3194
$\rho_{g,\tilde{R}}$	0.3411	0.5499	1.6123
$\rho_{\tau_b,\tilde{R}}$	0.1200	1.2617	10.5144
$\rho_{\tilde{R}}$	0.4412	0.1101	0.2495
q_{11}	0.0110	0.0006	0.0579
q_{21}	0.0037	0.0009	0.2430
q_{31}	0.0058	0.0024	0.4065
q_{41}	0.0009	0.0013	1.4043
q_{51}	0.0005	0.0100	19.9724
q_{61}	0.0003	0.0003	1.1280
q_{22}	0.0092	0.0007	0.0717
q_{32}	-0.0008	0.0028	3.5125
q_{42}	0.0050	0.0014	0.2832
q_{52}	-0.0175	0.0189	1.0819
q_{62}	0.0000	0.0003	0.0003
q_{33}	0.0029	0.0022	0.7422
q_{43}	0.0117	0.0050	0.4255
q_{53}	-0.0013	0.0237	18.2111
q_{63}	0.0001	0.0033	32.6605
q_{44}	0.0087	0.0067	0.7715
q_{54}	0.0014	0.0292	20.8730
q_{64}	-0.0002	0.0045	22.4300
q_{55}	0.0219	0.0116	0.5293
q_{65}	0.0040	0.0005	0.1246
q_{66}	0.0010	0.0012	1.1923

Table 15: MBCA, Parameter Identification (Some Deep Parameters Free)

	value	*CRLB*	*rCRLB*
z_{ss}	-0.0246	2.1912	88.9226
$\tau_{l_{ss}}$	0.1267	0.4744	3.7431
$\tau_{x_{ss}}$	0.4649	6.2711	13.4892
g_{ss}	-1.5389	0.0694	0.0451
$\tilde{R}_{ss}$	0.0000	0.0023	0.0023
ρ_z	0.8541	0.5182	0.6068
$\rho_{\tau_l,z}$	-0.0437	0.3059	6.9991
$\rho_{\tau_x,z}$	-0.0557	0.1738	3.1195
$\rho_{g,z}$	0.0533	0.3181	5.9690
$\rho_{\tau_b,z}$	0.0633	0.7185	11.3585
$\rho_{\tilde{R},z}$	-0.0141	0.0880	6.2419
ρ_{z,τ_l}	-0.1482	0.4659	3.1440
ρ_{τ_l}	1.0580	0.3470	0.3280
ρ_{τ_x,τ_l}	-0.0335	0.9381	28.0039
ρ_{g,τ_l}	0.0587	0.2177	3.7081
ρ_{τ_b,τ_l}	-0.2979	1.2796	4.2953
$\rho_{\tilde{R},\tau_l}$	0.0146	0.1146	7.8510
ρ_{z,τ_x}	0.2654	0.4307	1.6228
ρ_{τ_l,τ_x}	-0.0014	0.4689	334.9210
ρ_{τ_x}	1.0877	0.5609	0.5157
ρ_{g,τ_x}	-0.0974	0.2500	2.5668
ρ_{τ_b,τ_x}	0.0850	1.1771	13.8482
$\rho_{\tilde{R},\tau_x}$	0.0005	0.0920	184.0737
$\rho_{z,g}$	-0.0094	0.5517	58.5039
$\rho_{\tau_l,g}$	0.0097	0.1280	13.1959
$\rho_{\tau_x,g}$	0.0026	0.2396	92.1551
ρ_g	1.0053	0.2162	0.2151
$\rho_{\tau_b,g}$	-0.0076	0.2403	31.6135
$\rho_{\tilde{R},g}$	0.0004	0.0235	58.7368
ρ_{z,τ_b}	-0.0654	0.5120	7.8303
ρ_{τ_l,τ_b}	0.0465	0.0947	2.0375
ρ_{τ_x,τ_b}	-0.0116	0.6453	55.6283
ρ_{g,τ_b}	0.0241	0.1851	7.6786
ρ_{τ_b}	0.8263	0.5197	0.6289
$\rho_{\tilde{R},\tau_b}$	0.0063	0.0362	5.7450
$\rho_{z,\tilde{R}}$	0.7994	2.0086	2.5127
$\rho_{\tau_l,\tilde{R}}$	-0.7219	0.5524	0.7652
$\rho_{\tau_x,\tilde{R}}$	0.4016	7.3077	18.1964
$\rho_{g,\tilde{R}}$	0.3411	0.7627	2.2359
$\rho_{\tau_b,\tilde{R}}$	0.1200	3.9863	33.2192
$\rho_{\tilde{R}}$	0.4412	0.1855	0.4204
q_{11}	0.0110	0.0046	0.4202
q_{21}	0.0037	0.0012	0.3225
q_{31}	0.0058	0.0188	3.2388
q_{41}	0.0009	0.0014	1.5813
q_{51}	0.0005	0.0187	37.4840
q_{61}	0.0003	0.0005	1.6802
q_{22}	0.0092	0.0061	0.6675
q_{32}	-0.0008	0.0189	23.6018
q_{42}	0.0050	0.0015	0.2959
q_{52}	-0.0175	0.0225	1.2844
q_{62}	0.0000	0.0004	0.0004
q_{33}	0.0029	0.0202	6.9657
q_{43}	0.0117	0.0815	6.9617
q_{53}	-0.0013	0.1007	77.4485
q_{63}	0.0001	0.0112	111.8444
q_{44}	0.0087	0.1096	12.5977
q_{54}	0.0014	0.0854	61.0234
q_{64}	-0.0002	0.0184	92.0608
q_{55}	0.0219	0.0165	0.7539
q_{65}	0.0040	0.0029	0.7225
q_{66}	0.0010	0.0022	2.2075
g_n	0.0037	0.0923	24.9454
g_z	0.0040	0.0112	2.8109
δ	0.0118	0.1014	8.5917
σ	1.0000	2.7391	2.7391
α	0.3500	0.3514	1.0039
ρ_R	0.7500	0.2378	0.3170
ω_π	1.5000	1.8906	1.2604

Table 16: MBCA, Information Matrix Decomposition (Observing 6 Standard Variables)

	CRLB/para.	sens/para.	coll.	ϱ_i	$\varrho_{i(1)}$	$\varrho_{i(2)}$
z_{ss}	5.411	0.437	12.388	0.996736	0.923 ($\tau_{x_{ss}}$)	0.945 ($\tau_{x_{ss}}$, $\tilde{R}_{ss}$)
$\tau_{l_{ss}}$	0.272	0.079	3.436	0.956717	0.451 (g_{ss})	0.630 (z_{ss}, g_{ss})
$\tau_{x_{ss}}$	0.282	0.017	16.867	0.998241	0.923 (z_{ss})	0.955 (z_{ss}, $\tilde{R}_{ss}$)
g_{ss}	0.038	0.009	4.104	0.969858	0.736 ($\tau_{x_{ss}}$)	0.850 (z_{ss}, $\tau_{l_{ss}}$)
$\tilde{R}_{ss}$	0.002	0.000	5.086	0.980482	0.496 ($\tau_{x_{ss}}$)	0.676 (z_{ss}, $\tau_{x_{ss}}$)
ρ_z	0.332	0.001	420.889	0.999997	0.993 (ρ_{z,τ_x})	0.999 (ρ_{z,τ_x}, $\rho_{z,g}$)
$\rho_{\tau_l,z}$	3.442	0.018	187.343	0.999986	0.989 (ρ_{τ_l,τ_x})	0.997 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
$\rho_{\tau_x,z}$	2.501	0.005	456.118	0.999998	0.992 (ρ_{τ_x})	0.998 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
$\rho_{g,z}$	3.329	0.021	159.247	0.999980	0.990 (ρ_{g,τ_x})	0.997 (ρ_{g,τ_x}, ρ_g)
$\rho_{\tau_b,z}$	6.654	0.035	191.861	0.999986	0.979 (ρ_{τ_b,τ_x})	0.989 (ρ_{τ_b,τ_x}, $\rho_{\tau_b,g}$)
$\rho_{\tilde{R},z}$	1.623	0.017	94.732	0.999944	0.990 ($\rho_{\tilde{R},\tau_x}$)	0.996 ($\rho_{\tilde{R},\tau_x}$, $\rho_{\tilde{R},g}$)
ρ_{z,τ_l}	2.755	0.014	194.002	0.999987	0.917 (ρ_{τ_x,τ_l})	0.982 (ρ_{τ_x,τ_l}, ρ_{τ_b,τ_l})
ρ_{τ_l}	0.199	0.002	118.928	0.999965	0.790 (ρ_{τ_b,τ_l})	0.965 (ρ_{τ_l,τ_x}, ρ_{τ_l,τ_b})
ρ_{τ_x,τ_l}	4.864	0.024	201.703	0.999988	0.949 (ρ_{g,τ_l})	0.986 (ρ_{z,τ_l}, ρ_{τ_b,τ_l})
ρ_{g,τ_l}	3.644	0.045	80.899	0.999924	0.949 (ρ_{τ_x,τ_l})	0.975 (ρ_{g,τ_x}, ρ_{g,τ_b})
ρ_{τ_b,τ_l}	2.328	0.013	185.382	0.999985	0.790 (ρ_{τ_l})	0.939 (ρ_{τ_b,τ_x}, ρ_{τ_b})
$\rho_{\tilde{R},\tau_l}$	2.018	0.045	44.361	0.999746	0.675 ($\rho_{\tilde{R},g}$)	0.931 ($\rho_{\tilde{R},\tau_x}$, $\rho_{\tilde{R},\tau_b}$)
ρ_{z,τ_x}	1.372	0.003	435.327	0.999997	0.993 (ρ_z)	0.999 (ρ_z, $\rho_{z,g}$)
ρ_{τ_l,τ_x}	146.307	0.696	210.110	0.999989	0.989 ($\rho_{\tau_l,z}$)	0.997 ($\rho_{\tau_l,z}$, $\rho_{\tau_l,g}$)
ρ_{τ_x}	0.169	0.000	488.457	0.999998	0.992 ($\rho_{\tau_x,z}$)	0.998 ($\rho_{\tau_x,z}$, $\rho_{\tau_x,g}$)
ρ_{g,τ_x}	2.288	0.014	163.487	0.999981	0.990 ($\rho_{g,z}$)	0.998 ($\rho_{g,z}$, ρ_g)
ρ_{τ_b,τ_x}	7.296	0.032	231.052	0.999991	0.979 ($\rho_{\tau_b,z}$)	0.990 ($\rho_{\tau_b,z}$, $\rho_{\tau_b,g}$)
$\rho_{\tilde{R},\tau_x}$	71.695	0.605	118.481	0.999964	0.990 ($\rho_{\tilde{R},z}$)	0.996 ($\rho_{\tilde{R},z}$, $\rho_{\tilde{R},g}$)
$\rho_{z,g}$	11.492	0.081	141.658	0.999975	0.929 ($\rho_{\tau_x,g}$)	0.988 ($\rho_{\tau_x,g}$, $\rho_{\tau_b,g}$)
$\rho_{\tau_l,g}$	6.377	0.072	88.102	0.999936	0.824 ($\rho_{\tau_b,g}$)	0.921 ($\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$)
$\rho_{\tau_x,g}$	12.535	0.120	104.264	0.999954	0.937 (ρ_g)	0.989 ($\rho_{z,g}$, $\rho_{\tau_b,g}$)
ρ_g	0.048	0.001	47.038	0.999774	0.937 ($\rho_{\tau_x,g}$)	0.952 ($\rho_{z,g}$, $\rho_{\tau_b,g}$)
$\rho_{\tau_b,g}$	29.644	0.222	133.747	0.999972	0.824 ($\rho_{\tau_l,g}$)	0.920 ($\rho_{z,g}$, $\rho_{\tau_x,g}$)
$\rho_{\tilde{R},g}$	21.888	0.659	33.218	0.999547	0.675 ($\rho_{\tilde{R},\tau_l}$)	0.820 ($\rho_{\tilde{R},z}$, $\rho_{\tilde{R},\tau_x}$)
ρ_{z,τ_b}	2.567	0.014	180.589	0.999985	0.918 (ρ_{τ_x,τ_b})	0.982 (ρ_{τ_x,τ_b}, ρ_{τ_b})
ρ_{τ_l,τ_b}	1.861	0.017	107.845	0.999957	0.767 (ρ_{τ_b})	0.967 (ρ_{τ_l}, ρ_{τ_l,τ_x})
ρ_{τ_x,τ_b}	6.577	0.031	212.675	0.999989	0.966 (ρ_{g,τ_b})	0.986 (ρ_{z,τ_b}, ρ_{g,τ_b})
ρ_{g,τ_b}	3.902	0.049	78.892	0.999920	0.966 (ρ_{τ_x,τ_b})	0.977 (ρ_{g,τ_l}, ρ_{g,τ_x})
ρ_{τ_b}	0.328	0.002	165.624	0.999982	0.767 (ρ_{τ_l,τ_b})	0.947 (ρ_{τ_b,τ_l}, ρ_{τ_b,τ_x})
$\rho_{\tilde{R},\tau_b}$	2.172	0.046	47.425	0.999778	0.528 ($\rho_{\tilde{R},z}$)	0.932 ($\rho_{\tilde{R},\tau_l}$, $\rho_{\tilde{R},\tau_x}$)
$\rho_{z,\tilde{R}}$	0.997	0.038	25.915	0.999255	0.889 ($\rho_{\tau_x,\tilde{R}}$)	0.953 ($\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$)
$\rho_{\tau_l,\tilde{R}}$	0.631	0.039	16.073	0.998063	0.500 ($\rho_{\tilde{R}}$)	0.871 ($\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$)
$\rho_{\tau_x,\tilde{R}}$	1.319	0.038	34.906	0.999590	0.899 ($\rho_{g,\tilde{R}}$)	0.969 ($\rho_{z,\tilde{R}}$, $\rho_{g,\tilde{R}}$)
$\rho_{g,\tilde{R}}$	1.612	0.164	9.860	0.994844	0.899 ($\rho_{\tau_x,\tilde{R}}$)	0.924 ($\rho_{z,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$)
$\rho_{\tau_b,\tilde{R}}$	10.514	0.367	28.685	0.999392	0.481 ($\rho_{\tilde{R}}$)	0.868 ($\rho_{\tau_l,\tilde{R}}$, $\rho_{\tilde{R}}$)
$\rho_{\tilde{R}}$	0.249	0.016	15.685	0.997965	0.846 (q_{55})	0.941 ($\rho_{\tilde{R},z}$, q_{55})
q_{11}	0.058	0.016	3.584	0.960297	0.937 (q_{31})	0.960 (q_{21}, q_{31})
q_{21}	0.243	0.050	4.858	0.978583	0.939 (q_{51})	0.968 (q_{11}, q_{51})
q_{31}	0.407	0.021	19.169	0.998638	0.937 (q_{11})	0.950 (q_{11}, q_{41})
q_{41}	1.404	0.720	1.951	0.858718	0.840 (q_{31})	0.842 (q_{21}, q_{31})
q_{51}	19.972	0.870	22.955	0.999051	0.970 (q_{61})	0.979 (q_{21}, q_{61})
q_{61}	1.128	0.273	4.131	0.970260	0.970 (q_{51})	0.970 (q_{31}, q_{51})
q_{22}	0.072	0.020	3.675	0.962262	0.912 (q_{52})	0.939 (q_{42}, q_{52})
q_{32}	3.512	0.154	22.840	0.999041	0.840 (q_{42})	0.884 (ρ_{τ_b,τ_l}, q_{42})
q_{42}	0.283	0.129	2.187	0.889370	0.840 (q_{32})	0.842 (q_{22}, q_{32})
q_{52}	1.082	0.025	43.524	0.999736	0.970 (q_{62})	0.976 (q_{22}, q_{62})
q_{62}	0.000	0.000	4.237	0.971743	0.970 (q_{52})	0.970 (q_{32}, q_{52})
q_{33}	0.742	0.038	19.698	0.998711	0.745 (q_{43})	0.786 ($\rho_{\tau_l,g}$, q_{43})
q_{43}	0.426	0.055	7.692	0.991514	0.745 (q_{33})	0.756 ($\rho_{\tau_l,g}$, q_{33})
q_{53}	18.211	0.335	54.425	0.999831	0.970 (q_{63})	0.973 ($\rho_{g,\tilde{R}}$, q_{63})
q_{63}	32.661	0.819	39.874	0.999685	0.970 (q_{53})	0.973 ($\rho_{g,\tilde{R}}$, q_{53})
q_{44}	0.772	0.055	13.997	0.997445	0.515 ($\rho_{\tau_l,g}$)	0.580 ($\rho_{z,g}$, $\rho_{\tau_x,g}$)
q_{54}	20.873	0.311	67.113	0.999889	0.970 (q_{64})	0.976 ($\rho_{g,\tilde{R}}$, q_{64})
q_{64}	22.430	0.410	54.717	0.999833	0.970 (q_{54})	0.976 ($\rho_{g,\tilde{R}}$, q_{54})
q_{55}	0.529	0.019	27.447	0.999336	0.943 (q_{65})	0.956 ($\rho_{\tilde{R}}$, q_{65})
q_{65}	0.125	0.020	6.091	0.986429	0.943 (q_{55})	0.943 ($\rho_{\tau_l,\tilde{R}}$, q_{55})
q_{66}	1.192	0.058	20.582	0.998819	0.242 ($\rho_{\tilde{R}}$)	0.454 ($\rho_{\tilde{R}}$, q_{55})

Table 17: MBCA (Some Deep Parameters Estimated), Information Matrix Decomposition

	CRLB/para.	sens/para.	coll.	ϱ_i	$\varrho_{i(1)}$	$\varrho_{i(2)}$
z_{ss}	88.923	0.437	203.582	0.999988	0.967 (α)	0.985 (δ, α)
$\tau_{l_{ss}}$	3.743	0.079	47.283	0.999776	0.451 (g_{ss})	0.675 (g_{ss}, α)
$\tau_{x_{ss}}$	13.489	0.017	807.496	0.999999	0.981 (g_n)	0.990 (g_n, δ)
g_{ss}	0.045	0.009	4.807	0.978125	0.736 ($\tau_{x_{ss}}$)	0.853 ($\tau_{l_{ss}}$, α)
$\tilde{R}_{ss}$	0.002	0.000	5.282	0.981918	0.566 (g_n)	0.835 (g_n, α)
ρ_z	0.607	0.001	769.983	0.999999	0.993 (ρ_{z,τ_x})	0.999 (ρ_{z,τ_x}, $\rho_{z,g}$)
$\rho_{\tau_l,z}$	6.999	0.018	380.905	0.999997	0.989 (ρ_{τ_l,τ_x})	0.997 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
$\rho_{\tau_x,z}$	3.120	0.005	568.821	0.999998	0.992 (ρ_{τ_x})	0.998 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
$\rho_{g,z}$	5.969	0.021	285.575	0.999994	0.990 (ρ_{g,τ_x})	0.997 (ρ_{g,τ_x}, ρ_g)
$\rho_{\tau_b,z}$	11.358	0.035	327.492	0.999995	0.979 (ρ_{τ_b,τ_x})	0.989 (ρ_{τ_b,τ_x}, $\rho_{\tau_b,g}$)
$\rho_{\tilde{R},z}$	6.242	0.017	364.272	0.999996	0.990 ($\rho_{\tilde{R},\tau_x}$)	0.996 ($\rho_{\tilde{R},\tau_x}$, $\rho_{\tilde{R},g}$)
ρ_{z,τ_l}	3.144	0.014	221.371	0.999990	0.917 (ρ_{τ_x,τ_l})	0.982 (ρ_{τ_x,τ_l}, ρ_{τ_b,τ_l})
ρ_{τ_l}	0.328	0.002	195.729	0.999987	0.790 (ρ_{τ_b,τ_l})	0.965 (ρ_{τ_l,τ_x}, ρ_{τ_l,τ_b})
ρ_{τ_x,τ_l}	28.004	0.024	1161.364	1.000000	0.949 (ρ_{g,τ_l})	0.986 (ρ_{z,τ_l}, ρ_{τ_b,τ_l})
ρ_{g,τ_l}	3.708	0.045	82.311	0.999926	0.949 (ρ_{τ_x,τ_l})	0.975 (ρ_{g,τ_x}, ρ_{g,τ_b})
ρ_{τ_b,τ_l}	4.295	0.013	342.062	0.999996	0.790 (ρ_{τ_l})	0.939 (ρ_{τ_b,τ_x}, ρ_{τ_b})
$\rho_{\tilde{R},\tau_l}$	7.851	0.045	172.608	0.999983	0.675 ($\rho_{\tilde{R},g}$)	0.931 ($\rho_{\tilde{R},\tau_x}$, $\rho_{\tilde{R},\tau_b}$)
ρ_{z,τ_x}	1.623	0.003	514.749	0.999998	0.993 (ρ_z)	0.999 (ρ_z, $\rho_{z,g}$)
ρ_{τ_l,τ_x}	334.921	0.696	480.977	0.999998	0.989 ($\rho_{\tau_l,z}$)	0.997 ($\rho_{\tau_l,z}$, $\rho_{\tau_l,g}$)
ρ_{τ_x}	0.516	0.000	1489.071	1.000000	0.992 ($\rho_{\tau_x,z}$)	0.998 ($\rho_{\tau_x,z}$, $\rho_{\tau_x,g}$)
ρ_{g,τ_x}	2.567	0.014	183.404	0.999985	0.990 ($\rho_{g,z}$)	0.998 ($\rho_{g,z}$, ρ_g)
ρ_{τ_b,τ_x}	13.848	0.032	438.524	0.999997	0.979 ($\rho_{\tau_b,z}$)	0.990 ($\rho_{\tau_b,z}$, $\rho_{\tau_b,g}$)
$\rho_{\tilde{R},\tau_x}$	184.074	0.605	304.196	0.999995	0.990 ($\rho_{\tilde{R},z}$)	0.996 ($\rho_{\tilde{R},z}$, $\rho_{\tilde{R},g}$)
$\rho_{z,g}$	58.504	0.081	721.174	0.999999	0.929 ($\rho_{\tau_x,g}$)	0.988 ($\rho_{\tau_x,g}$, $\rho_{\tau_b,g}$)
$\rho_{\tau_l,g}$	13.196	0.072	182.309	0.999985	0.824 ($\rho_{\tau_b,g}$)	0.921 ($\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$)
$\rho_{\tau_x,g}$	92.155	0.120	766.509	0.999999	0.937 (ρ_g)	0.989 ($\rho_{z,g}$, $\rho_{\tau_b,g}$)
ρ_g	0.215	0.001	212.213	0.999989	0.937 ($\rho_{\tau_x,g}$)	0.952 ($\rho_{z,g}$, $\rho_{\tau_b,g}$)
$\rho_{\tau_b,g}$	31.613	0.222	142.631	0.999975	0.824 ($\rho_{\tau_l,g}$)	0.920 ($\rho_{z,g}$, $\rho_{\tau_x,g}$)
$\rho_{\tilde{R},g}$	58.737	0.659	89.143	0.999937	0.675 ($\rho_{\tilde{R},\tau_l}$)	0.820 ($\rho_{\tilde{R},z}$, $\rho_{\tilde{R},\tau_x}$)
ρ_{z,τ_b}	7.830	0.014	550.781	0.999998	0.918 (ρ_{τ_x,τ_b})	0.982 (ρ_{τ_x,τ_b}, ρ_{τ_b})
ρ_{τ_l,τ_b}	2.038	0.017	118.095	0.999964	0.767 (ρ_{τ_b})	0.967 (ρ_{τ_l}, ρ_{τ_l,τ_x})
ρ_{τ_x,τ_b}	55.628	0.031	1798.754	1.000000	0.966 (ρ_{g,τ_b})	0.986 (ρ_{z,τ_b}, ρ_{g,τ_b})
ρ_{g,τ_b}	7.679	0.049	155.247	0.999979	0.966 (ρ_{τ_x,τ_b})	0.977 (ρ_{g,τ_l}, ρ_{g,τ_x})
ρ_{τ_b}	0.629	0.002	317.345	0.999995	0.767 (ρ_{τ_l,τ_b})	0.947 (ρ_{τ_b,τ_l}, ρ_{τ_b,τ_x})
$\rho_{\tilde{R},\tau_b}$	5.745	0.046	125.447	0.999968	0.528 ($\rho_{\tilde{R},z}$)	0.932 ($\rho_{\tilde{R},\tau_l}$, $\rho_{\tilde{R},\tau_x}$)
$\rho_{z,\tilde{R}}$	2.513	0.038	65.329	0.999883	0.889 ($\rho_{\tau_x,\tilde{R}}$)	0.953 ($\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$)
$\rho_{\tau_l,\tilde{R}}$	0.765	0.039	19.483	0.998682	0.500 ($\rho_{\tilde{R}}$)	0.871 ($\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$)
$\rho_{\tau_x,\tilde{R}}$	18.196	0.038	481.414	0.999998	0.899 ($\rho_{g,\tilde{R}}$)	0.969 ($\rho_{z,\tilde{R}}$, $\rho_{g,\tilde{R}}$)
$\rho_{g,\tilde{R}}$	2.236	0.164	13.674	0.997322	0.899 ($\rho_{\tau_x,\tilde{R}}$)	0.924 ($\rho_{z,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$)
$\rho_{\tau_b,\tilde{R}}$	33.219	0.367	90.628	0.999939	0.593 (ρ_R)	0.868 ($\rho_{\tau_l,\tilde{R}}$, $\rho_{\tilde{R}}$)
$\rho_{\tilde{R}}$	0.420	0.016	26.429	0.999284	0.850 (ρ_R)	0.950 (q_{55}, ω_π)
q_{11}	0.420	0.016	26.007	0.999260	0.937 (q_{31})	0.960 (q_{21}, q_{31})
q_{21}	0.323	0.050	6.449	0.987903	0.939 (q_{51})	0.968 (q_{11}, q_{51})
q_{31}	3.239	0.021	152.717	0.999979	0.937 (q_{11})	0.950 (q_{11}, q_{41})
q_{41}	1.581	0.720	2.197	0.890440	0.840 (q_{31})	0.842 (q_{21}, q_{31})
q_{51}	37.484	0.870	43.082	0.999731	0.970 (q_{61})	0.979 (q_{21}, q_{61})
q_{61}	1.680	0.273	6.153	0.986706	0.970 (q_{51})	0.970 (q_{51}, ω_π)
q_{22}	0.668	0.020	34.190	0.999572	0.912 (q_{52})	0.939 (q_{42}, q_{52})
q_{32}	23.602	0.154	153.473	0.999979	0.840 (q_{42})	0.884 (ρ_{τ_b,τ_l}, q_{42})
q_{42}	0.296	0.129	2.285	0.899189	0.840 (q_{32})	0.842 (q_{22}, q_{32})
q_{52}	1.284	0.025	51.670	0.999813	0.970 (q_{62})	0.976 (q_{22}, q_{62})
q_{62}	0.000	0.000	4.512	0.975134	0.970 (q_{52})	0.970 (q_{32}, q_{52})
q_{33}	6.966	0.038	184.862	0.999985	0.745 (q_{43})	0.786 ($\rho_{\tau_l,g}$, q_{43})
q_{43}	6.962	0.055	125.846	0.999968	0.745 (q_{33})	0.756 ($\rho_{\tau_l,g}$, q_{33})
q_{53}	77.448	0.335	231.461	0.999991	0.970 (q_{63})	0.973 ($\rho_{g,\tilde{R}}$, q_{63})
q_{63}	111.844	0.819	136.547	0.999973	0.970 (q_{53})	0.973 ($\rho_{g,\tilde{R}}$, q_{53})
q_{44}	12.598	0.055	228.544	0.999990	0.515 ($\rho_{\tau_l,g}$)	0.627 ($\rho_{\tau_l,g}$, σ)
q_{54}	61.023	0.311	196.210	0.999987	0.970 (q_{64})	0.976 ($\rho_{g,\tilde{R}}$, q_{64})
q_{64}	92.061	0.410	224.580	0.999990	0.970 (q_{54})	0.976 ($\rho_{g,\tilde{R}}$, q_{54})
q_{55}	0.754	0.019	39.094	0.999673	0.943 (q_{65})	0.956 ($\rho_{\tilde{R}}$, q_{65})
q_{65}	0.722	0.020	35.319	0.999599	0.943 (q_{55})	0.952 (q_{55}, ρ_R)
q_{66}	2.208	0.058	38.106	0.999656	0.535 (ω_π)	0.729 (ρ_R, ω_π)
g_n	24.945	0.037	666.842	0.999999	0.981 ($\tau_{x_{ss}}$)	0.988 ($\tau_{x_{ss}}$, $\rho_{g,z}$)
g_z	2.811	0.120	23.506	0.999095	0.961 (δ)	0.967 (g_n, δ)
δ	8.592	0.039	222.782	0.999990	0.961 (g_z)	0.979 (z_{ss}, g_z)
σ	2.739	0.016	174.691	0.999984	0.557 (z_{ss})	0.723 (z_{ss}, q_{44})
α	1.004	0.003	370.406	0.999996	0.974 ($\tau_{x_{ss}}$)	0.990 (z_{ss}, $\tau_{x_{ss}}$)
ρ_R	0.317	0.006	52.055	0.999815	0.850 ($\rho_{\tilde{R}}$)	0.902 ($\rho_{\tau_l,\tilde{R}}$, $\rho_{\tilde{R}}$)
ω_π	1.260	0.020	63.537	0.999876	0.738 (ρ_R)	0.886 (q_{66}, ρ_R)

Table 18: MBCA Model, Information Matrix Decomposition (Observing 6 Wedges)

	CRLB/para.	sens/para.	coll.	ϱ_i	$\varrho_{i(1)}$	$\varrho_{i(2)}$
z_{ss}	5.486	0.099	55.563	0.999838	0.861 ($\tau_{x_{ss}}$)	0.998 ($\tau_{x_{ss}}$, $\tau_{b_{ss}}$)
$\tau_{l_{ss}}$	0.308	0.016	18.980	0.998611	0.977 ($\tau_{b_{ss}}$)	0.995 (g_{ss}, $\tau_{b_{ss}}$)
$\tau_{x_{ss}}$	0.233	0.004	64.877	0.999881	0.904 (g_{ss})	0.998 (z_{ss}, $\tau_{b_{ss}}$)
g_{ss}	0.045	0.007	6.672	0.988704	0.904 ($\tau_{x_{ss}}$)	0.914 (z_{ss}, $\tau_{b_{ss}}$)
$\tau_{b_{ss}}$	0.080	0.004	20.423	0.998801	0.977 ($\tau_{l_{ss}}$)	0.996 ($\tau_{l_{ss}}$, g_{ss})
$\tilde{R}_{ss}$	0.002	0.000	6.817	0.989181	0.838 ($\tau_{l_{ss}}$)	0.908 (z_{ss}, $\tau_{x_{ss}}$)
ρ_z	0.095	0.001	113.826	0.999961	0.994 (ρ_{z,τ_x})	0.999 (ρ_{z,τ_x}, $\rho_{z,g}$)
$\rho_{\tau_l,z}$	1.691	0.017	102.092	0.999952	0.994 (ρ_{τ_l,τ_x})	0.999 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
$\rho_{\tau_x,z}$	0.875	0.009	102.811	0.999953	0.994 (ρ_{τ_x})	0.999 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
$\rho_{g,z}$	2.179	0.048	45.056	0.999754	0.991 (ρ_{g,τ_x})	0.999 (ρ_{g,τ_x}, ρ_g)
$\rho_{\tau_b,z}$	3.307	0.027	122.830	0.999967	0.994 (ρ_{τ_b,τ_x})	0.999 (ρ_{τ_b,τ_x}, $\rho_{\tau_b,g}$)
$\rho_{\tilde{R},z}$	2.189	0.023	95.026	0.999945	0.994 ($\rho_{\tilde{R},\tau_x}$)	0.999 ($\rho_{\tilde{R},\tau_x}$, $\rho_{\tilde{R},g}$)
ρ_{z,τ_l}	0.575	0.016	35.237	0.999597	0.952 (ρ_{τ_x,τ_l})	0.977 (ρ_{τ_l}, ρ_{τ_x,τ_l})
ρ_{τ_l}	0.072	0.002	31.555	0.999498	0.937 (ρ_{τ_b,τ_l})	0.977 (ρ_{τ_l,τ_x}, ρ_{τ_l,τ_b})
ρ_{τ_x,τ_l}	1.511	0.049	30.748	0.999471	0.952 (ρ_{z,τ_l})	0.976 (ρ_{τ_x}, ρ_{τ_x,τ_b})
ρ_{g,τ_l}	2.055	0.127	16.157	0.998083	0.858 (ρ_g)	0.982 (ρ_{g,τ_x}, ρ_{g,τ_b})
ρ_{τ_b,τ_l}	0.726	0.019	37.927	0.999652	0.963 ($\rho_{\tilde{R},\tau_l}$)	0.977 (ρ_{τ_b,τ_x}, ρ_{τ_b})
$\rho_{\tilde{R},\tau_l}$	2.191	0.074	29.561	0.999428	0.963 (ρ_{τ_b,τ_l})	0.976 ($\rho_{\tilde{R},\tau_x}$, $\rho_{\tilde{R},\tau_b}$)
ρ_{z,τ_x}	0.399	0.003	118.786	0.999965	0.994 (ρ_z)	0.999 (ρ_z, $\rho_{z,g}$)
ρ_{τ_l,τ_x}	68.331	0.642	106.404	0.999956	0.994 ($\rho_{\tau_l,z}$)	0.999 ($\rho_{\tau_l,z}$, $\rho_{\tau_l,g}$)
ρ_{τ_x}	0.058	0.001	106.998	0.999956	0.994 ($\rho_{\tau_x,z}$)	0.999 ($\rho_{\tau_x,z}$, $\rho_{\tau_x,g}$)
ρ_{g,τ_x}	1.542	0.033	46.985	0.999773	0.991 ($\rho_{g,z}$)	0.999 ($\rho_{g,z}$, ρ_g)
ρ_{τ_b,τ_x}	3.188	0.025	128.102	0.999970	0.994 ($\rho_{\tau_b,z}$)	0.999 ($\rho_{\tau_b,z}$, $\rho_{\tau_b,g}$)
$\rho_{\tilde{R},\tau_x}$	79.866	0.807	98.976	0.999949	0.994 ($\rho_{\tilde{R},z}$)	0.999 ($\rho_{\tilde{R},z}$, $\rho_{\tilde{R},g}$)
$\rho_{z,g}$	1.514	0.089	16.932	0.998255	0.953 ($\rho_{\tau_x,g}$)	0.977 ($\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$)
$\rho_{\tau_l,g}$	1.310	0.087	15.082	0.997800	0.937 ($\rho_{\tau_b,g}$)	0.964 ($\rho_{z,g}$, $\rho_{\tau_b,g}$)
$\rho_{\tau_x,g}$	3.269	0.221	14.792	0.997712	0.953 ($\rho_{z,g}$)	0.967 ($\rho_{z,g}$, $\rho_{\tau_b,g}$)
ρ_g	0.020	0.002	8.356	0.992814	0.858 (ρ_{g,τ_l})	0.947 ($\rho_{g,z}$, ρ_{g,τ_x})
$\rho_{\tau_b,g}$	4.759	0.262	18.154	0.998482	0.963 ($\rho_{\tilde{R},g}$)	0.974 ($\rho_{\tau_l,g}$, $\rho_{\tilde{R},g}$)
$\rho_{\tilde{R},g}$	13.415	0.945	14.195	0.997515	0.963 ($\rho_{\tau_b,g}$)	0.967 ($\rho_{\tilde{R},\tau_l}$, $\rho_{\tau_b,g}$)
ρ_{z,τ_b}	0.628	0.016	38.536	0.999663	0.952 (ρ_{τ_x,τ_b})	0.979 (ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b})
ρ_{τ_l,τ_b}	0.792	0.023	33.771	0.999561	0.935 (ρ_{τ_b})	0.977 ($\rho_{\tau_l,z}$, ρ_{τ_l})
ρ_{τ_x,τ_b}	2.103	0.061	34.475	0.999579	0.952 (ρ_{z,τ_b})	0.979 ($\rho_{\tau_x,z}$, ρ_{τ_x,τ_l})
ρ_{g,τ_b}	2.409	0.144	16.731	0.998212	0.845 (ρ_{τ_x,τ_b})	0.981 ($\rho_{g,z}$, ρ_{g,τ_l})
ρ_{τ_b}	0.126	0.003	40.518	0.999695	0.968 ($\rho_{\tilde{R},\tau_b}$)	0.977 ($\rho_{\tau_b,z}$, ρ_{τ_b,τ_l})
$\rho_{\tilde{R},\tau_b}$	2.442	0.077	31.533	0.999497	0.968 (ρ_{τ_b})	0.977 ($\rho_{\tilde{R},z}$, $\rho_{\tilde{R},\tau_l}$)
$\rho_{z,\tilde{R}}$	0.380	0.037	10.192	0.995175	0.953 ($\rho_{\tau_x,\tilde{R}}$)	0.979 ($\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$)
$\rho_{\tau_l,\tilde{R}}$	0.379	0.042	9.006	0.993817	0.937 ($\rho_{\tau_b,\tilde{R}}$)	0.966 ($\rho_{z,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$)
$\rho_{\tau_x,\tilde{R}}$	0.450	0.050	9.012	0.993824	0.953 ($\rho_{z,\tilde{R}}$)	0.968 ($\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$)
$\rho_{g,\tilde{R}}$	1.255	0.312	4.017	0.968518	0.836 ($\rho_{\tau_x,\tilde{R}}$)	0.865 ($\rho_{g,z}$, $\rho_{\tau_x,\tilde{R}}$)
$\rho_{\tau_b,\tilde{R}}$	6.461	0.595	10.849	0.995743	0.968 ($\rho_{\tilde{R}}$)	0.976 ($\rho_{\tau_l,\tilde{R}}$, $\rho_{\tilde{R}}$)
$\rho_{\tilde{R}}$	0.259	0.031	8.453	0.992978	0.968 ($\rho_{\tau_b,\tilde{R}}$)	0.969 ($\rho_{\tilde{R},z}$, $\rho_{\tau_b,\tilde{R}}$)
q_{11}	0.058	0.016	3.585	0.960317	0.937 (q_{31})	0.960 (q_{21}, q_{31})
q_{21}	0.212	0.050	4.231	0.971666	0.939 (q_{51})	0.968 (q_{11}, q_{51})
q_{31}	0.072	0.021	3.389	0.955473	0.937 (q_{11})	0.950 (q_{11}, q_{41})
q_{41}	1.404	0.719	1.952	0.858840	0.840 (q_{31})	0.842 (q_{21}, q_{31})
q_{51}	4.598	0.870	5.286	0.981945	0.970 (q_{61})	0.979 (q_{21}, q_{61})
q_{61}	1.128	0.273	4.131	0.970265	0.970 (q_{51})	0.970 (q_{31}, q_{51})
q_{22}	0.058	0.020	2.964	0.941350	0.912 (q_{52})	0.939 (q_{42}, q_{52})
q_{32}	0.302	0.154	1.967	0.861055	0.840 (q_{42})	0.861 (q_{22}, q_{42})
q_{42}	0.246	0.129	1.899	0.850068	0.840 (q_{32})	0.842 (q_{22}, q_{32})
q_{52}	0.118	0.025	4.746	0.977547	0.970 (q_{62})	0.976 (q_{22}, q_{62})
q_{62}	0.000	0.000	4.127	0.970205	0.970 (q_{52})	0.970 (q_{32}, q_{52})
q_{33}	0.058	0.038	1.538	0.759661	0.746 (q_{43})	0.759 (q_{43}, q_{53})
q_{43}	0.084	0.055	1.520	0.752922	0.746 (q_{33})	0.752 (q_{33}, q_{63})
q_{53}	1.383	0.334	4.133	0.970290	0.970 (q_{63})	0.970 (q_{33}, q_{63})
q_{63}	3.379	0.819	4.127	0.970198	0.970 (q_{53})	0.970 (q_{43}, q_{53})
q_{44}	0.058	0.055	1.052	0.310266	0.307 (q_{54})	0.309 (q_{54}, q_{64})
q_{54}	1.282	0.311	4.126	0.970180	0.970 (q_{64})	0.970 (q_{44}, q_{64})
q_{64}	1.689	0.410	4.124	0.970155	0.970 (q_{54})	0.970 (q_{44}, q_{54})
q_{55}	0.058	0.019	2.993	0.942520	0.943 (q_{65})	0.943 (ρ_g, q_{65})
q_{65}	0.061	0.020	2.993	0.942515	0.943 (q_{55})	0.943 (ρ_g, q_{55})
q_{66}	0.058	0.058	1.000	0.007684	0.001 (ρ_{τ_l})	0.002 (ρ_{τ_l}, $\rho_{\tau_l,g}$)

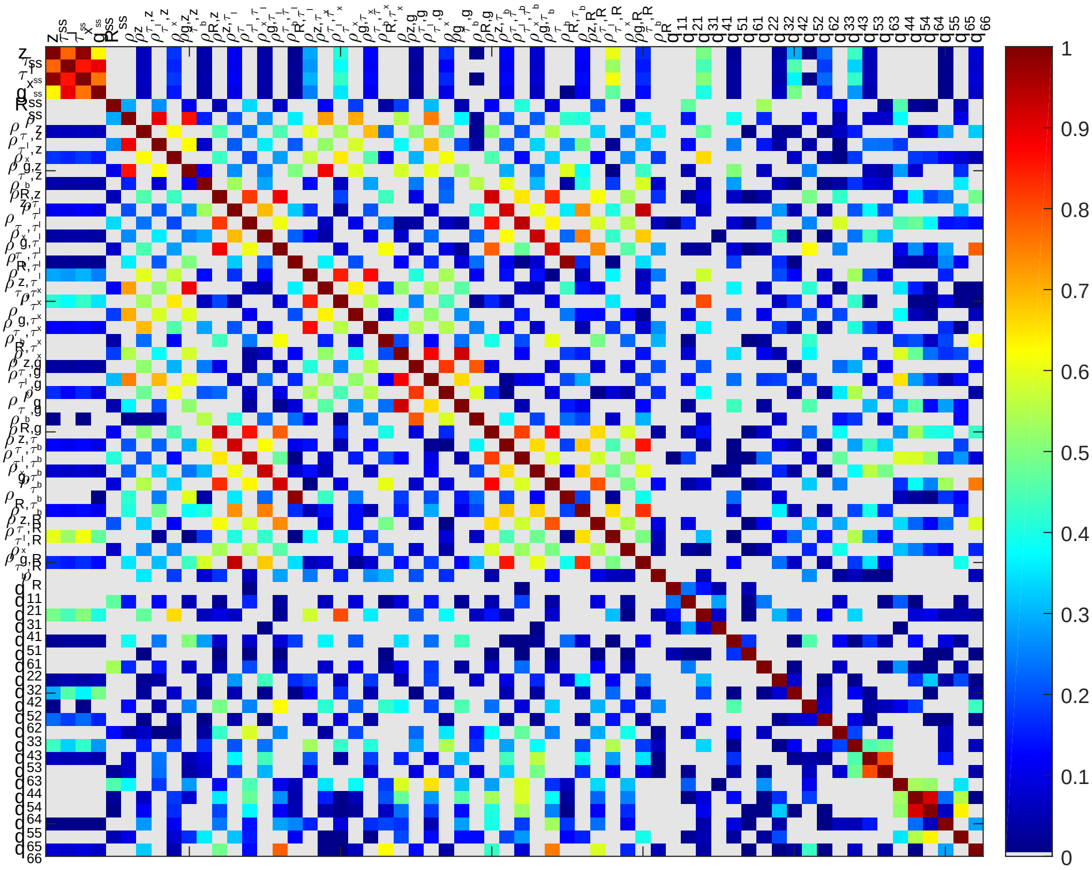

Figure 3: MBCA, Pairwise Correlations (Observing 6 Standard Variables)

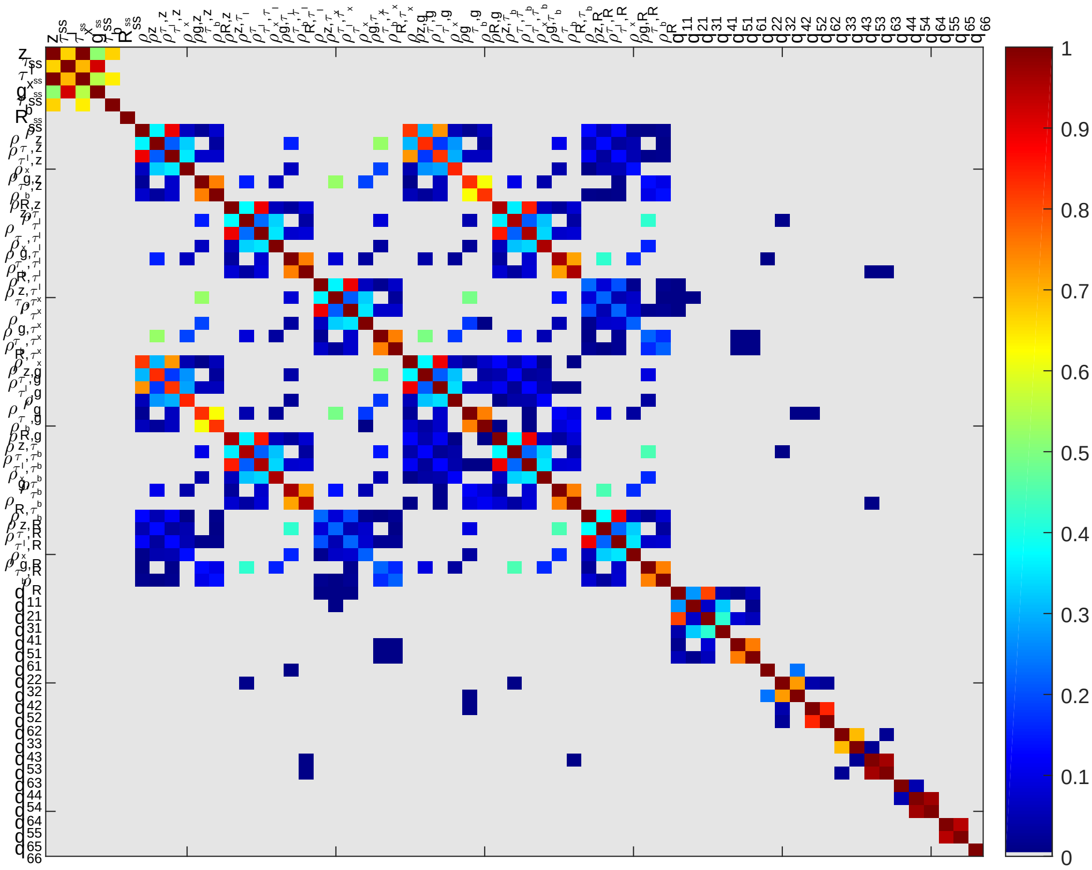

Figure 4: MBCA, Pairwise Correlations (Observing 6 Wedges)

5 Economic Relevance

To assess which wedge is most important in accounting for cyclical fluctuations in the data Brinca et al. (2016) use the statistic f_i^Y given by

$$f_i^Y = \frac{1/\sum_t (Y_t - Y_{it})^2}{\sum_{t,j}(1/(Y_t - Y_{jt})^2)},$$

where $j = \{z,\ \tau_l,\ \tau_x,\ g\}$ in the BCA and $j = \{z,\ \tau_l,\ \tau_x,\ g,\ \tau_b,\ \tilde{R}\}$ in the monetary BCA model. Y_t is actual data of a given observable $Y = \{y, l, x\}$ (output, labor or investment)[5] and Y_{it} is the component of observable Y due to wedge i and with $f_i^Y \in [0, 1]$, $\sum f_i^Y = 1$. For instance, f_i^x roughly measures the fraction of movement and level in actual investment explained by wedge i. The statistic f_i^Y ranges from 0 to 1 and is increasing in the explanatory power of a wedge. Indeed, when $Y_t = Y_{it}$, then $f_i^Y = 1$ since $f_j^Y = 0\ \forall j \neq i$ whereas, in the limit, $f_i^Y = 0$ when the deviation of actual data from its component due to a given wedge goes to infinity. Notice that since the MSE statistic is used a model can be penalized along both the variance and the bias dimension. In other words both missing the data by a constant and missing the variation in the data is penalized.

When simulating the observables, we follow Chari et al. (2007) and consider two classes of counterfactual economies, namely the "one-wedge-on" and "one-wedge-off" economies. These economies are constructed by feeding the estimated wedges back into the model either one at a time ("one-wedge-on") or all but one ("one wedge-off"). It is important to point out the fact that the wedges have both a direct and a forecasting effect on the model. In the experiments, we seek to retain the forecasting effect only. This is done by setting the inactive wedges equal to some constant value (typically their intercept) at time t while preserving the estimated stochastic process and the realization of the wedges at time $t - 1$ to forecast their future realizations. Notice that this procedure would not be necessary if the matrices P and Q in the VAR(1) law of motion of the wedge shocks were diagonal.

The intuition behind the "one-wedge-on" and "one-wedge-off" counterfactual economies is the following. On one hand, the first seeks to understand how far a single wedge channel can bring the model to replicate the movements in the data while turning off the other margins. On the other hand, the second asks how badly the model performs when freezing the same channel and while keeping the others active.

5.1 Chari et al. (2007) BCA Model

Table 19 shows the f_i^Y statistic for different observables and counterfactual economies as well as its one standard error bands given by f_i +/- Sd(f_i), where the standard deviations are computed using the delta method and the parameter covariance matrix (inverse Fisher information matrix) for the case when only the wedges parameters are assumed unknown (and thus the deep parameters of the model are fixed). To calculate the statistics we focus on the 1982 recession episode as in Chari et al. (2007) and use their original data.

The one standard deviation bands around f_i statistics for the one-wedge-on counterfactual economies cover an economically non-negligible range. However, these bands never overlap and are always quite far from each other. This means that if one used this statistic to measure and rank the relative importance of the wedges in replicating movements in

[5] We use Chari et al. (2007) actual data for the study of the 1982 recession.

the data, the main conclusions would not be overturned[6]. Thus, the main result of Chari et al. (2007) still holds trough: The labor and the efficiency wedge play primary roles in explaining business cycle fluctuations during the period covering 1982 recession whereas the investment and government wedge are negligible.

This is no longer true if one considers the one-wedge-off economies since the statistics exhibit such a higher level of uncertainty that the bands around them overlap. Our analysis thus discourages from using this statistic only to evaluate the relative importance of the wedges in a BCA model.

The intuition behind this result is that while the one-wedge-on f_i statistics are a function of only one object subject to uncertainty, the one-wedge-off f_i statistics are a function of three such objects. Indeed, in the one-wedge-on counterfactual economies only one wedge is active whereas in the one-wedge-off experiments this is the case for three wedges.

Table 19: BCA Model, Uncertainty Around f_i-Statistics

Observable	**Counterfactual Economy**	f_i-Sd(f_i)	f_i	f_i-Sd(f_i)
Output	Efficiency Wedge On	0.62676	**0.77628**	0.9258
	Labor Wedge On	0.049776	0.12629	0.20281
	Investment Wedge On	2.4019e-05	0.044326	0.088627
	Government Wedge On	0.010541	0.053098	0.095654
Hours	Efficiency Wedge On	-0.070856	0.062223	0.1953
	Labor Wedge On	0.69063	**0.89148**	1.0923
	Investment Wedge On	-0.01805	0.02228	0.06261
	Government Wedge On	-0.013868	0.024022	0.061912
Investment	Efficiency Wedge On	0.57444	**0.6803**	0.78617
	Labor Wedge On	0.068189	0.1912	0.31421
	Investment Wedge On	-0.023585	0.055209	0.134
	Government Wedge On	0.044278	0.073286	0.10229
Output	Efficiency Wedge Off	-0.013808	0.038301	0.09041
	Labor Wedge Off	-0.073664	0.19775	0.46917
	Investment Wedge Off	0.10669	**0.4555**	0.80432
	Government Wedge Off	-0.0011315	0.30844	0.61802
Hours	Efficiency Wedge Off	0.43456	**0.89559**	1.3566
	Labor Wedge Off	-0.057498	0.022289	0.10208
	Investment Wedge Off	-0.17741	0.049314	0.27604
	Government Wedge Off	-0.13141	0.032808	0.19702
Investment	Efficiency Wedge Off	-0.019469	0.084061	0.18759
	Labor Wedge Off	-0.05529	0.28093	0.61716
	Investment Wedge Off	0.053152	0.13215	0.21114
	Government Wedge Off	0.072597	**0.50286**	0.93312

5.2 Šustek (2011) Monetary BCA Model

Forthcoming.

[6] Testing that the f_i statistics are not significantly different from each other would be the most rigorous way of answering this question.

6 Statistics for Practitioners

In this section we present some statistics that might be of special interest to (Monetary) Business Cycle Accounting practitioners. We show how the size of the sample available affects the overall strength of parameter identification.

6.1 Empirical Distance Measures

While the analysis in the previous sections has been carried out in the time domain using the state space representation of the DSGE model solution the methodology proposed by Qu and Tkachenko (2012) and Qu and Tkachenko (2016) takes a frequency domain perspective to study identification issues. In particular, rank conditions for local identification of dynamic parameters are established by studying the spectral density matrix which maps the structural parameters to functions in the Banach space. Local identifiability occurs if and only if the a local change in the parameter values leads to a different image. This analysis thus requires to study the Jacobian of the spectral density matrix w.r.t. the deep parameters of the model. If steady state related parameters are estimated as well, like in our case, it is possible to incorporate in the analysis also the first order properties of the model by considering an extra term involving the steady state parameters. In both cases, under some regularity conditions, the rank conditions are necessary and sufficient for local identification also in the time domain. More specifically, it can be shown that when the model is nonsingular, like in our case, analyzing the rank conditions in the frequency domain is equivalent to inspecting the rank of the information matrix in the time domain. This analysis is thus expected to deliver analogous results to Iskrev (2015) and Komunjer and Ng (2011). For a more detailed comparison of these tests we refer the reader to Qu and Tkachenko (2012).

In general, the solution of a DSGE model log-linearized around its steady state can be expressed in state space form. Assuming no measurement errors, as it is the case in the BCA and MBCA model, it is given by

$$\begin{aligned} X_{t+1} &= A(\theta)X_t + B(\theta)\varepsilon_{t+1}, \\ Y_t &= \widehat{C}(\theta)X_t \end{aligned} \tag{6.1}$$

and can be rewritten as a Vector Moving Average (VMA) representation using

$$\begin{aligned} Y_t(\theta) &= \widehat{C}(\theta)A(\theta)X_{t-1} + B(\theta)\varepsilon_t \\ &= \widehat{C}(\theta)A(\theta)\left(A(\theta)X_{t-2} + B(\theta)\varepsilon_{t-1}\right) + B(\theta)\varepsilon_t \\ &= \widehat{C}(\theta)A(\theta)^2X_{t-2} + \widehat{C}(\theta)A(\theta)B(\theta)\varepsilon_{t-1} + B(\theta)\varepsilon_t \\ &= \dots \end{aligned} \tag{6.2}$$

where, following the exposition in Qu and Tkachenko (2012) we made explicit the dependence of Y_t on θ. More generally the process can be expressed as

$$Y_t(\theta) = \widehat{C}(\theta)A(\theta)^kX_{t-k} + \widehat{C}(\theta)A(\theta)^{k-1}B(\theta)\varepsilon_{t-k+1} + \ldots + \widehat{C}(\theta)A(\theta)B(\theta)\varepsilon_{t-1} + B(\theta)\varepsilon_t. \tag{6.3}$$

If all eigenvalues of $A(\theta)$ then $\lim_{k\to\infty} A(\theta)^k X_{t-k} = 0$. It thus follows that

$$Y_t(\theta) = \sum_{j=0}^{\infty} \widehat{C}(\theta)A(\theta)^j B(\theta)\varepsilon_{t-j} \tag{6.4}$$

$$= \sum_{j=0}^{\infty} h_j(\theta)\varepsilon_{t-j} \tag{6.5}$$

$$= H(L;\theta)\varepsilon_t, \tag{6.6}$$

where $h_j(\theta)$ $(j = 0, ..., \infty)$ are $n_Y \times n_\varepsilon$ matrices and $H(L;\theta) = \sum_{j=0}^{\infty} h_j(\theta)L^j$ the matrix of lagged polynomials.
Equation 6.5 along with some standard assumption about the process of the error term ε implies that $Y_t(\theta)$ is covariance stationary and has a spectral density matrix $f_\theta(\omega)$ that can be represented as

$$f_\theta(\omega) = \frac{1}{2\pi} H(\exp(-i\omega);\theta)\Sigma(\theta)H(\exp(-i\omega);\theta)^*, \tag{6.7}$$

where X^* denotes the conjugate transpose of a generic complex matrix X.

In some cases, like in the BCA and MBCA methodological framework, not only dynamic but also steady state parameters are estimated. In this case, it is useful to define the augmented parameter vector as $\bar{\theta} = (\theta', \alpha')'$ and make explicit the relationship between data observables Y_t^o, model observables $Y_t(\theta)$ and steady states $\mu(\bar{\theta})$

$$Y_t^o = \mu(\bar{\theta}) + Y_t(\theta), \tag{6.8}$$

where it is important to recall the fact that the model observables are expressed as log-deviations form their respective steady state values. This representation is useful in the case where steady state parameters are estimated since considering only the second order properties would omit the full identification potential of these parameters. This is why, in this case, the identification of $\bar{\theta}$ will be examined on the properties of $\mu(\bar{\theta})$ and $f_{\bar{\theta}}(\omega)$ jointly.

Definition 6.1. The parameter vector $\bar{\theta}$ is said to be locally identifiable from the first and second order properties of $\{Y_t\}$ at $\bar{\theta} = \bar{\theta}_0$ if there exists an open neighborhood of $\bar{\theta}_0$ in which $\mu(\bar{\theta}_1) = \mu(\bar{\theta}_0)$ and $f_{\bar{\theta}_1}(\omega) = f_{\bar{\theta}_0}(\omega)$ for all $\omega \in [-\pi, \pi]$ necessarily implies $\bar{\theta}_1 = \bar{\theta}_0$.

Now consider the following object

$$\bar{G}(\bar{\theta}) = \int_{-\pi}^{\pi} \left(\frac{\partial \text{vec}(f_\theta(\omega))}{\bar{\theta}'}\right)' \left(\frac{\partial \text{vec}(f_\theta(\omega))}{\bar{\theta}'}\right) d\omega + \left(\frac{\partial \mu(\bar{\theta})}{\partial \bar{\theta}'}\right)' \left(\frac{\partial \mu(\bar{\theta})}{\partial \bar{\theta}'}\right). \tag{6.9}$$

Under some mild assumptions Qu and Tkachenko (2012) show that

Theorem 6.1. *$\bar{\theta}$ is locally identifiable from the first and second order properties of Y_t^o at a point $\bar{\theta}_0$ if and only if $\bar{G}(\bar{\theta}_0)$ is nonsingular. Establishing whether the parameter set θ is identifiable thus reduces to checking that the matrix $\bar{G}(\bar{\theta}_0)$ is full rank.*

Analogously to Komunjer and Ng (2011) the eigenvalues of $G(\bar{\theta}_0)$ are obtained numerically, with the sole exception of the default Matlab tolerance level being used to determine its rank. This is due to the fact that the matrix is not as sparse as the one considered by Komunjer and Ng (2011). Also, the derivatives $\frac{\partial f_\theta(\omega_j)}{\theta_k}$ are computed numerically as $[f_{\theta_0+\mathbf{e}_k h_k}(\omega_j) - f_{\theta_0}(\omega_j)]/$ where $\mathbf{e}_k$ is a unit vector whose k-th element is equal to one and h_k is the step size. Unlike Komunjer and Ng (2011), however, the step size is not set to 1e-3 but to 1e-7.

To estimate $\frac{\partial f_\theta(\omega_j)}{\theta_k}$ and to compute the integral in 6.9 we make use of use the symmetric difference quotient and Gaussian quadrature respectively. We find that both models are locally identifiable only when the first and second order properties of the data are used since we find the matrix $\bar{G}(\bar{\theta}_0)$ in (6.9) to be full rank but the matrix $\bar{G}(\bar{\theta}) = \int_{-\pi}^{\pi} \left(\frac{\partial \mathrm{vec}(f_\theta(\omega))}{\theta'}\right)' \left(\frac{\partial \mathrm{vec}(f_\theta(\omega))}{\theta'}\right) d\omega)$ to be rank deficient.

If $\bar{\theta}$ is shown to be locally unidentifiable, as it is the case here for both the BCA and MBCA model, it is possible to trace out a curve of points χ which are observationally equivalent to $\bar{\theta}_0$ in a local neighborhood of the latter. Indeed Qu and Tkachenko (2012), following Rothenberg (1971), show that this curve can be defined using the function $\theta(v)$ such that

$$\frac{\partial \bar{\theta}(v)}{\partial v} = c(\bar{\theta}),\ \bar{\theta}(0) = \bar{\theta}_0, \tag{6.10}$$

where $c(\bar{\theta}_0)$ is the eigenvector which corresponds to the smallest eigenvalue of $\bar{G}(\bar{\theta})$, v is a scalar which varies in a neighborhood of 0 such that $\bar{\theta}(v) \in \delta(\bar{\theta}_0)$[7].

As shown by Qu and Tkachenko (2012), along the nonidentification curve χ the parameter set $\bar{\theta}$ is not identified at $\bar{\theta}_0$ since $\frac{\partial \bar{\theta}(v)}{\partial v} = 0$ and thus $c(\bar{\theta}) = 0$, which implies

$$\frac{\partial \mathrm{vec}(f_{\bar{\theta}(v)}(\omega))}{\partial v} = \frac{\partial \mathrm{vec}(f_{\bar{\theta}(v)}(\omega))}{\partial \bar{\theta}(v)'} c(\bar{\theta}) = 0 \tag{6.11}$$

for all $\omega \in [-\pi, \pi]$.

The curve can be traced out recursively using the Euler method such that $\bar{\theta}(v_{j+1}) = \bar{\theta}(v_j) + c(\bar{\theta}(v_j))h$, where h is the step size (fixed at 1e-04).

We apply two methods (the first due to Qu and Tkachenko (2012) and the latter to Qu and Tkachenko (2016)) can be used to robustify conclusions on both local and global identification success or failure.

The first method recognizes the fact that it is important to make sure that the points on the curve result in identical spectral densities. It thus computes the maximum absolute and relative deviations of $f_{\bar{\theta}_1}(\omega)$ from $f_{\bar{\theta}_0}(\omega)$ across the frequencies: $\max_{\omega\in[0,\pi]} ||f_{\bar{\theta}}(\omega) - f_{\bar{\theta}_0}(\omega)||_\infty$[8] and $\left\{\max_{\omega\in[0,\pi]} ||f_{\bar{\theta}}(\omega) - f_{\bar{\theta}_0}(\omega)||_\infty\right\} \backslash f_{\bar{\theta}_0,hl}(\omega)$, where $f_{\bar{\theta}_0,hl}(\omega)$ denotes the (h,l) element of the spectral density matrix given the parameter set θ and is evaluated at the same frequency and element of $f_{\bar{\theta}_0}(\omega)$ that maximizes the numerator. In the BCA model we find that at norm($\bar{\theta}_0 - \bar{\theta}$)=0.1276 the maximum absolute deviation on the curve is 0.0015 whereas the maximum relative deviation in relative form 1.3811e-04. When

[7]As pointed out by Qu and Tkachenko (2012) it is usually the case that $\delta(\bar{\theta}_0)$ is unknown and, thus, so is the domain of the curve. This is why in constructing the nonidentification curve first a wide support is considered, the model solved and the spectrum computed. The resulting curve is then truncated so as to exclude points which i) are associated with indeterminacy, ii) violate the natural bounds of the parameters and iii) yield $f_{\bar{\theta}}(\omega)$ different from $f_{\bar{\theta}_0}(\omega)$.

[8]There is no need to consider $\omega \in [-\pi, 0]$ because $f_{\bar{\theta}}(\omega)$ is equal to the conjugate of $f_{\bar{\theta}}(-\omega)$.

norm($\bar\theta_0 - \bar\theta$)= is 1.30 the two measures are 0.0083 and 7.5130e-04 respectively. As to the monetary BCA model, when norm($\bar\theta_0 - \bar\theta$)=0.0251 the maximum absolute deviation on the curve is 1.0226e-04 and the maximum relative deviation in relative form 3.2755e-05. When norm($\bar\theta_0 - \bar\theta$)=2.1728 the two measures are 2.3858e-04 and 7.4645e-05 respectively. In both models, the points on the curve $\bar\theta_0$ for the smallest norm are chosen such that at least the absolute measure is bigger than the numerical errors associated with the Euler method, which Qu and Tkachenko (2016) say should remain near or below 1.0E-04.

The second method computes a so-called "empirical distance between DSGE models" using a range of sample sizes. It is equal to the testing power of the likelihood ratio test of the null hypothesis that $f_{\bar\theta}(\omega)$ is the true spectral density against the alternative hypothesis that $h_{\phi_0}(\omega)$ is the true spectral density. In the context of our analysis, $h_{\phi_0}(\omega)$ is taken to be $f_{\bar\theta_0}(\omega)$ but the test is general enough to accommodate different parameter vectors as well as different model structures. The empirical distance measure has the following properties. First, its value is between 0 and 1 for any sample size T and significance level α. A higher value thus means that it is easier to distinguish between the spectral densities implied by the two parameter vectors $\bar\theta$ and $\bar\theta_0$ and thus argues in favour of stronger identification. Second, consider what Qu and Tkachenko (2016) refer to as the "Kullback-Leibler distance between two DSGE models (and, more generally, between two vector linear processes) with spectral densities $f_{\bar\theta_0}(\omega)$ and $f_{\bar\theta_0}(\omega)$". It is given by

$$KL(\bar\theta_0, \bar\theta_1) = KL(\theta_0, \theta_1) + \frac{1}{4\pi}\left(\mu(\bar\theta_0) - \mu(\bar\theta_1)\right)' f_{\bar\theta_1}^{-1}(0)\left(\mu(\bar\theta_0) - \mu(\bar\theta_1)\right)' \quad (6.12)$$

$$\text{where} \quad KL(\theta_0, \theta_1) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\mathrm{tr}(f_{\bar\theta_1}^{-1}(\omega) f_{\bar\theta_0}(\omega)) - \log\det(f_{\bar\theta_1}^{-1}(\omega) f_{\bar\theta_0}(\omega)) - n_Y\right\} d\omega \quad (6.13)$$

To provide intuition, let $l(\bar\theta)$ denote the frequency domain approximate likelihood (or the time domain Gaussian likelihood) based on $Y_t(\bar\theta)$. Then, $T^{-1}\mathbb{E}_{Y_t(\bar\theta),\bar\theta=\bar\theta_0}\left(L(\bar\theta_0) - L(\bar\theta_1)\right) \to KL(\bar\theta_0, \bar\theta_1)$. If the Kullback-Leibler distance $KL(\bar\theta, \bar\theta_0)$ is nonzero its value increase consistently with T and approaches 1 as $T \to \infty$. As becomes evident from Table 20 and Table 21 all empirical distance measures (p-values of the likelihood ratio tests) are well above the 5% significance level for all sample sizes considered, even for a small sample size of just $T = 20$, i.e. 5 years of observations, and for the smallest normed differences between the points on the identification curve and the points at which local identification is checked in the respective models. This is reassuring evidence for practitioners since even with a relatively small sample size the overall identification strength is sufficiently high.

Table 20: BCA Model, Wedges Parameters, Empirical Distance Measures

Empirical distance measures on the curve when norm($\bar\theta_0 - \bar\theta$)=0.1276							
Sample Size	$T = 20$	$T = 40$	$T = 80$	$T = 120$	$T = 160$	$T = 200$	$T = 1000$
	0.1211	0.1654	0.2438	0.3155	0.3820	0.4437	0.9570
Empirical distance measures on the curve when norm($\bar\theta_0 - \bar\theta$)=1.30							
Sample Size	$T = 20$	$T = 40$	$T = 80$	$T = 120$	$T = 160$	$T = 200$	$T = 1000$
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 21: MBCA Model, Wedges Parameters (no $\tau_{b_{ss}}$, $\tilde{R}_{ss}$), Empirical Distance Measures

Empirical distance measures on the curve when norm($\bar{\theta}_0 - \bar{\theta}$)=0.0251							
Sample Size	$T = 20$	$T = 40$	$T = 80$	$T = 120$	$T = 160$	$T = 200$	$T = 1000$
	0.2517	0.3955	0.6204	0.7719	0.8675	0.9251	1.0000
Empirical distance measures on the curve when norm($\bar{\theta}_0 - \bar{\theta}$)=2.1728							
Sample Size	$T = 20$	$T = 40$	$T = 80$	$T = 120$	$T = 160$	$T = 200$	$T = 1000$
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

7 Conclusion

In the past years, Business Cycle Accounting exercises have sparked great interest among theoretical and applied macroeconomists insofar as they can shed light on which classes of models are able to explain fluctuations in macroeconomic aggregates during a particular economic episode. These exercises involve maximum likelihood estimation of the stochastic process governing the latent variables. Even though they have been extensively performed, the methodology has yet to be properly scrutinized in terms of identification deficiencies. Given that quantitative recommendations are made the statistical and economic relevance of potential identification issues of the methodology is of the utmost importance.

In this paper we take seriously the notes of caution raised by the literature which investigates identification issues in DSGE models. Indeed, we first perform strict and weak local identification tests as theorized by Komunjer and Ng (2011) and Iskrev (2015) on the parameters vectors estimated in Chari et al. (2007) and Šustek (2011). We find that in both the standard and monetary BCA frameworks the model parameters are strictly identifiable. This is no longer true once one extends the estimation to the deep parameters of the model. In these cases we show how to obviate such failures by imposing restrictions on the space of estimated parameters. Our analysis also points out that both models are affected by weak identification problems and thus suffer from a low degree of estimation precision. In particular, we find that the elements which suffer from weak identification are the off-diagonal elements of the VAR(1) law of motion of the latent variables. This is due to the fact that while these parameters do affect the likelihood, the effect which they exert on the latter is strongly collinear. We find that this is an inherent property of the VAR(1) process evaluated at the estimated parameter vectors. Indeed, we show that when innovations to the wedges are assumed to be observed and the parameters are estimated using the VAR in isolation from the model it is the off-diagonal elements which have a strongly collinear effect on the likelihood. Second, we investigate the economic severity of these identification problems by computing the uncertainty around a statistic which is used to rank which classes of models best explain business cycle fluctuations. We find that the main conclusion are not overturned for the standard BCA model. We are currently investigating whether this results also holds for the monetary BCA framework. Finally, we investigate a question of interest to practitioners, namely how identification strength varies across sample sizes. We show that even in small samples the parameter sets in both frameworks are overall well identified by computing empirical distance measures as in Qu and Tkachenko (2016).

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A Appendix - BCA and MBCA Model with Investment Adjustment Costs

A.1 Komunjer and Ng (2011)

A.1.1 Chari et al. (2007) BCA Model

We introduce the version of the BCA model by Chari et al. (2007) which allows for investment adjustment costs calibrated at the “normal” intensity used by Bernanke et al. (1998), see Appendix C.2. The results are similar to the baseline case with the default parameters being strictly identifiable and several steady state wedge shocks, off-diagonal elements of the P and Q matrix as well as deep parameters not being identifiable once the set of estimated parameters is extended to the latter (Table A-1 and A-2).

In line with the analysis carried out for the baseline model, we check which restricted parameter combinations would allow the other parameters to be strictly identifiable. As reported in Table A-3 we find three such sets at the lowest tolerance level for which identification fails in A-2, mostly consisting of three parameters each. These sets are (i) $\{a,\ \rho_g\}$, (ii) $\{b,\ \rho_g\}$ and (iii) $\{a,\ \psi\}$ and thus involve the parameters governing the degree of investment adjustment costs (a and b), the autocorrelation of the government wedge and the inverse of the constant Frisch elasticity of labor supply ψ.

Table A-1: Komunjer and Ng Test Results BCA Model with Normal Adjustment Costs

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	29	25	15	51	40	62	0
e-03	29	25	16	53	45	68	0
e-04	29	25	16	54	45	69	0
e-05	29	25	16	54	45	69	0
e-06	29	25	16	54	45	69	0
e-07	29	25	16	54	45	69	0
e-08	30	25	16	54	46	70	0
e-09	30	25	16	54	46	70	0
e-10	30	25	16	55	46	70	0
e-11	30	25	16	55	46	71	1
Default=1.378453e-12	30	25	16	55	46	71	1
Required	30	25	16	55	46	71	1

Summary: $n_\theta = 30, n_X = 5, n_\varepsilon = 4$.

Order Condition: $n_\theta = 30, n_\delta = 50$.

Table A-2: Komunjer and Ng Test Results BCA Model with Normal Adjustment Costs (Deep Parameters Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	34	25	15	53	45	64	0
e-03	36	25	16	57	52	71	0
e-04	36	25	16	57	52	71	0
e-05	36	25	16	58	52	71	0
e-06	36	25	16	58	52	71	0
e-07	36	25	16	58	52	71	0
e-08	37	25	16	60	53	74	0
e-09	37	25	16	61	53	75	0
e-10	38	25	16	61	54	76	0
e-11	38	25	16	63	54	77	0
Default=1.378453e-12	38	25	16	63	54	78	0
Required	39	25	16	64	55	80	1

Summary: $n_\theta = 39, n_X = 5, n_\varepsilon = 4$.

Order Condition: $n_\theta = 39, n_\delta = 50$.

Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{g,τ_x}, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, q_{21}, q_{22}, q_{32}, q_{42}, q_{33}, q_{43}, β, ψ, σ, a, b,

Problematic Parameters at Tol=1.378453e-12: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, q_{21}, q_{31}, q_{22}, q_{32}, q_{42}, q_{33}, q_{43}, q_{44}, g_n, g_z, β, δ, ψ, σ, α, a, b,

Table A-3: Komunjer and Ng Conditional Test Results BCA Model with Normal Adjustment Costs, Tol = 1.378453e-12

Fixed	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
a ρ_g	39	25	16	64	55	80	1
b ρ_g	39	25	16	64	55	80	1
a ψ	39	25	16	64	55	80	1
Required	39	25	16	64	55	80	1

Summary: $n_\theta = 39, n_X = 5, n_\varepsilon = 4$.

Order Condition: $n_\theta = 39, n_\delta = 50$.

A.1.2 Šustek (2011) MBCA Model

An analogous pattern to the one found in the baseline MBCA model and reported in Section 4 emerges once investment adjustment costs are introduced in the model (see Tables A-4, A-8, A-5, A-6).

As to the conditional identification tests, we find that only for the MBCA model with normal adjustment costs where $\tau_{b_s s}$ and $\tilde{R}_{SS}$ not included in estimation it is possible to reparameterize the model in a way which makes the other parameters identifiable. We find 27 parameter combinations which restrict only four parameters at the lowest tolerance level for which identification fails in A-6 (see Table A-7). These sets mainly feature the steady state innovations to the efficiency and government wedge (z_{ss} and g_{ss}) as well as steady state inflation π_{ss} and some deep parameters. The parameter b which governs, with a, the degree of investment adjustment costs also shows up prominently in these sets. This is because this parameter is a function of other deep parameters and fixing it thus provides an additional restriction which can help to identify the latter.

Table A-4: Komunjer and Ng Test Results MBCA Model with Normal Adjustment Costs

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	60	49	33	101	89	130	0
e-03	60	49	36	107	96	142	0
e-04	60	49	36	109	96	144	0
e-05	60	49	36	109	96	144	0
e-06	60	49	36	109	96	144	0
e-07	60	49	36	109	96	145	0
e-08	60	49	36	109	96	145	0
e-09	61	49	36	109	97	145	0
e-10	61	49	36	110	97	145	0
e-11	61	49	36	110	97	146	1
Default=5.826450e-12	61	49	36	110	97	146	1
Required	61	49	36	110	97	146	1

Summary: $n_\theta = 61, n_X = 7, n_\varepsilon = 6$.
Order Condition: $n_\theta = 61, n_\delta = 105$.

Table A-5: Komunjer and Ng Test Results MBCA Model with Normal Adjustment Costs ($\tau_{b_{ss}}$ and $\tilde{R}_{ss}$ Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	62	49	33	103	91	132	0
e-03	62	49	36	109	98	143	0
e-04	62	49	36	111	98	146	0
e-05	62	49	36	111	98	146	0
e-06	62	49	36	111	98	146	0
e-07	62	49	36	111	98	147	0
e-08	62	49	36	111	98	147	0
Default=2.983143e-09	62	49	36	111	98	147	0
e-09	63	49	36	111	99	147	0
e-10	63	49	36	111	99	147	0
e-11	63	49	36	112	99	148	1
Required	63	49	36	112	99	148	1

Summary: $n_\theta = 63, n_X = 7, n_\varepsilon = 6$.

Order Condition: $n_\theta = 63, n_\delta = 105$.

Problematic Parameters at Tol=1e-3: z_{ss}, g_{ss},

Problematic Parameters at Tol=1.000000e-10: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{21}, q_{31}, q_{51}, q_{22}, q_{32}, q_{42}, q_{33}, q_{43}, q_{53}, q_{44}, q_{54}, q_{64}, q_{55},

Table A-6: Komunjer and Ng Test Results MBCA Model with Normal Adjustment Costs (Deep Parameters Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	69	49	35	104	99	133	0
e-03	70	49	36	113	106	146	0
e-04	70	49	36	116	106	148	0
e-05	70	49	36	117	106	148	0
e-06	70	49	36	117	106	148	0
e-07	71	49	36	118	107	149	0
e-08	71	49	36	119	107	152	0
e-09	72	49	36	119	108	153	0
e-10	73	49	36	120	109	154	0
Default=2.330580e-11	73	49	36	121	109	155	0
e-11	73	49	36	122	109	157	0
Required	74	49	36	123	110	159	1

Summary: $n_\theta = 74, n_X = 7, n_\varepsilon = 6$.

Order Condition: $n_\theta = 74, n_\delta = 105$.

Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{22}, q_{42}, q_{43}, q_{53}, g_n, g_z, β, ψ, σ, ρ_R, ω_π, ω_y, π_{ss}, a, b,

Problematic Parameters at Tol=1.000000e-11: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{11}, q_{21}, q_{31}, q_{51}, q_{61}, q_{22}, q_{32}, q_{42}, q_{52}, q_{62}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, q_{65}, q_{66}, g_n, g_z, β, δ, ψ, σ, α, ρ_R, ω_π, ω_y, π_{ss}, a, b,

Table A-7: Komunjer and Ng Conditional Test Results MBCA Model with Normal Adjustment Costs, Tol = 2.330580e-11

Fixed	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
z_{ss} $\tau_{l_{ss}}$ π_{ss} b	74	49	36	123	110	159	1
z_{ss} ρ_z π_{ss} b	74	49	36	123	110	159	1
z_{ss} ρ_{τ_l,τ_x} π_{ss} b	74	49	36	123	110	159	1
z_{ss} ρ_{τ_b} π_{ss} b	74	49	36	123	110	159	1
z_{ss} $\rho_{\tau_b,\tilde{R}}$ π_{ss} b	74	49	36	123	110	159	1
z_{ss} q_{31} π_{ss} b	74	49	36	123	110	159	1
z_{ss} q_{51} π_{ss} b	74	49	36	123	110	159	1
z_{ss} q_{54} π_{ss} b	74	49	36	123	110	159	1
z_{ss} ψ π_{ss} b	74	49	36	123	110	159	1
z_{ss} σ π_{ss} b	74	49	36	123	110	159	1
z_{ss} π_{ss} a b	74	49	36	123	110	159	1
$\tau_{l_{ss}}$ g_{ss} π_{ss} b	74	49	36	123	110	159	1
g_{ss} ρ_z π_{ss} b	74	49	36	123	110	159	1
g_{ss} ρ_{τ_l,τ_x} π_{ss} b	74	49	36	123	110	159	1
g_{ss} ρ_{τ_b} π_{ss} b	74	49	36	123	110	159	1
g_{ss} $\rho_{\tau_b,\tilde{R}}$ π_{ss} b	74	49	36	123	110	159	1
g_{ss} q_{31} π_{ss} b	74	49	36	123	110	159	1
g_{ss} q_{51} π_{ss} b	74	49	36	123	110	159	1
g_{ss} q_{54} π_{ss} b	74	49	36	123	110	159	1
g_{ss} ψ π_{ss} b	74	49	36	123	110	159	1
g_{ss} σ π_{ss} b	74	49	36	123	110	159	1
g_{ss} π_{ss} a b	74	49	36	123	110	159	1
ρ_z α π_{ss} b	74	49	36	123	110	158	0
$\rho_{\tau_b,\tilde{R}}$ α π_{ss} b	74	49	36	123	110	158	0
q_{54} α π_{ss} b	74	49	36	123	110	158	0
ψ α π_{ss} b	74	49	36	123	110	158	0
σ α π_{ss} b	74	49	36	123	110	158	0
Required	74	49	36	123	110	159	1

Summary: $n_\theta = 74, n_X = 7, n_\varepsilon = 6$.
Order Condition: $n_\theta = 74, n_\delta = 105$.

Table A-8: Komunjer and Ng Test Results MBCA Model with Normal Adjustment Costs ($\tau_{b_{ss}}$, $\tilde{R}_{ss}$ and Deep Parameters Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	71	49	35	105	101	134	0
e-03	72	49	36	114	108	146	0
e-04	72	49	36	118	108	149	0
e-05	72	49	36	119	108	150	0
e-06	72	49	36	119	108	150	0
e-07	73	49	36	120	109	151	0
Default=9.546056e-08	73	49	36	120	109	151	0
e-08	73	49	36	121	109	154	0
e-09	74	49	36	121	110	154	0
e-10	74	49	36	122	110	156	0
e-11	75	49	36	123	111	158	0
Required	76	49	36	125	112	161	1

Summary: $n_\theta = 76, n_X = 7, n_\varepsilon = 6.$

Order Condition: $n_\theta = 76, n_\delta = 105.$

Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{21}, q_{31}, q_{51}, q_{22}, q_{32}, q_{42}, q_{52}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, g_n, g_z, β, δ, ψ, σ, ρ_R, ω_π, ω_y, π_{ss}, a, b,

Problematic Parameters at Tol=1.000000e-11: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, $\tau_{b_{ss}}$, $\tilde{R}_{ss}$, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{11}, q_{21}, q_{31}, q_{41}, q_{51}, q_{61}, q_{22}, q_{32}, q_{42}, q_{52}, q_{62}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, q_{65}, q_{66}, g_n, g_z, β, δ, ψ, σ, α, ρ_R, ω_π, ω_y, π_{ss}, a, b,

A.2 Iskrev (2015)

A.2.1 Chari et al. (2007) BCA Model

We first study strict identification in the standard BCA model featuring investment adjustment costs of normal intensity. We start with the 38 parameters case (only a is considered in estimation since b is de facto a function of other structural parameters). Fixing g_n and g_z in the standard BCA case discussed above leaves 36 parameters and the rank of the information matrix is found to be 33. There are 7140 combinations of 33 parameters out of 36 and as many as 2406 combinations of parameters which, when fixed, deliver a rank of 33 and thus enable identification of the ones left unrestricted.

Table A-9: BCA Model with Normal Adjustment Costs, Parameter Identification (All Deep Parameters Fixed)

	value	$CRLB$	$rCRLB$
z_{ss}	-0.0239	0.0955	3.9926
$\tau_{l_{ss}}$	0.3279	0.1450	0.4423
$\tau_{x_{ss}}$	0.4834	0.2142	0.4431
g_{ss}	-1.5344	0.3212	0.2093
ρ_z	0.9800	0.0488	0.0498
$\rho_{\tau_l,z}$	-0.0330	0.0514	1.5588
$\rho_{\tau_x,z}$	-0.0702	0.1022	1.4548
$\rho_{z,g}$	0.0048	0.0596	12.3854
ρ_{z,τ_l}	-0.0138	0.0351	2.5492
ρ_{τ_l}	0.9564	0.0528	0.0552
ρ_{τ_x,τ_l}	-0.0460	0.1147	2.4941
ρ_{g,τ_l}	-0.0081	0.0470	5.7929
ρ_{z,τ_x}	-0.0117	0.0836	7.1323
ρ_{τ_l,τ_x}	-0.0451	0.0699	1.5497
ρ_{τ_x}	0.8962	0.0941	0.1050
ρ_{g,τ_x}	0.0488	0.0995	2.0369
$\rho_{z,g}$	0.0192	0.0802	4.1698
$\rho_{\tau_l,g}$	0.0569	0.0663	1.1650
$\rho_{\tau_x,g}$	0.1041	0.0949	0.9114
ρ_g	0.9711	0.0926	0.0953
q_{11}	0.0116	0.0007	0.0578
q_{21}	0.0014	0.0015	1.0792
q_{31}	-0.0105	0.0078	0.7389
q_{41}	-0.0006	0.0013	2.2903
q_{22}	0.0064	0.0004	0.0636
q_{32}	0.0010	0.0067	6.5323
q_{42}	0.0061	0.0050	0.8153
q_{33}	0.0158	0.0075	0.4712
q_{43}	0.0142	0.0023	0.1619
q_{44}	0.0046	0.0036	0.7903

Table A-9 is instructive about weak identification problems. Interestingly, relative uncertainty decreases such that $q31$ and $q44$ no longer feature among the set of worst identified parameters. This is no longer true for $q31$ once also the deep parameters of the model are considered in the identification analysis (see Table A-10). The same results of the no adjustment costs counterpart hold through, with the exception of α also being one of the most badly identified parameters.

Table A-10: BCA Model with Normal Adjustment Costs, Wedges Parameter Identification

	value	$CRLB$	$rCRLB$
z_{ss}	-0.0239	2.3406	97.8465
$\tau_{l_{ss}}$	0.3279	0.5711	1.7414
$\tau_{x_{ss}}$	0.4834	1.9206	3.9729
g_{ss}	-1.5344	0.7576	0.4937
ρ_z	0.9800	0.0554	0.0565
$\rho_{\tau_l,z}$	-0.0330	0.0545	1.6526
$\rho_{\tau_x,z}$	-0.0702	0.1190	1.6937
$\rho_{z,g}$	0.0048	0.0654	13.5906
ρ_{z,τ_l}	-0.0138	0.0548	3.9733
ρ_{τ_l}	0.9564	0.0586	0.0612
ρ_{τ_x,τ_l}	-0.0460	0.1305	2.8372
ρ_{g,τ_l}	-0.0081	0.0660	8.1379
ρ_{z,τ_x}	-0.0117	0.1096	9.3467
ρ_{τ_l,τ_x}	-0.0451	0.0876	1.9431
ρ_{τ_x}	0.8962	0.1101	0.1229
ρ_{g,τ_x}	0.0488	0.1026	2.1003
$\rho_{z,g}$	0.0192	0.0934	4.8530
$\rho_{\tau_l,g}$	0.0569	0.0795	1.3964
$\rho_{\tau_x,g}$	0.1041	0.1715	1.6475
ρ_g	0.9711	0.1042	0.1073
q_{11}	0.0116	0.0064	0.5536
q_{21}	0.0014	0.0035	2.4901
q_{31}	-0.0105	0.0163	1.5549
q_{41}	-0.0006	0.0013	2.2904
q_{22}	0.0064	0.0046	0.7156
q_{32}	0.0010	0.0113	10.9783
q_{42}	0.0061	0.0116	1.8940
q_{33}	0.0158	0.0244	1.5433
q_{43}	0.0142	0.0045	0.3185
q_{44}	0.0046	0.0045	0.9858
δ	0.0118	0.0015	0.1285
σ	1.0000	0.4395	0.4395
α	0.3500	0.4199	1.1996

Table A-11: BCA Model with Normal Adjustment Costs (Deep Parameters Estimated), Information Matrix Decomposition

	CRLB/para.	sens/para.	coll.	ϱ_i	$\varrho_{i(1)}$	$\varrho_{i(2)}$
z_{ss}	97.846	0.410	238.923	0.999991	0.926 (g_{ss})	0.980 (g_{ss}, α)
$\tau_{l_{ss}}$	1.741	0.019	90.135	0.999938	0.801 (α)	0.861 ($\tau_{x_{ss}}$, ρ_{τ_x})
$\tau_{x_{ss}}$	3.973	0.006	681.977	0.999999	0.946 (g_{ss})	0.983 (ρ_{g,τ_l}, α)
g_{ss}	0.494	0.003	151.290	0.999978	0.946 ($\tau_{x_{ss}}$)	0.989 (z_{ss}, α)
ρ_z	0.057	0.000	211.820	0.999989	0.992 ($\rho_{z,g}$)	0.994 ($\rho_{\tau_l,z}$, $\rho_{z,g}$)
$\rho_{\tau_l,z}$	1.653	0.038	43.825	0.999740	0.900 (ρ_{τ_l})	0.987 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
$\rho_{\tau_x,z}$	1.694	0.019	89.376	0.999937	0.948 ($\rho_{z,g}$)	0.985 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
$\rho_{z,g}$	13.591	0.074	184.275	0.999985	0.992 (ρ_z)	0.993 (ρ_z, $\rho_{\tau_x,z}$)
ρ_{z,τ_l}	3.973	0.012	336.677	0.999996	0.991 (ρ_{g,τ_l})	0.995 (ρ_{g,τ_l}, $\rho_{z,g}$)
ρ_{τ_l}	0.061	0.001	75.356	0.999912	0.983 ($\rho_{\tau_l,g}$)	0.992 (ρ_{τ_l,τ_x}, $\rho_{\tau_l,g}$)
ρ_{τ_x,τ_l}	2.837	0.018	155.079	0.999979	0.980 ($\rho_{\tau_x,g}$)	0.991 (ρ_{τ_x}, $\rho_{\tau_x,g}$)
ρ_{g,τ_l}	8.138	0.027	298.583	0.999994	0.991 (ρ_{z,τ_l})	0.994 (ρ_{z,τ_l}, ρ_g)
ρ_{z,τ_x}	9.347	0.010	979.814	0.999999	0.992 (ρ_{g,τ_x})	0.999 (ρ_z, $\rho_{z,g}$)
ρ_{τ_l,τ_x}	1.943	0.012	166.317	0.999982	0.987 ($\rho_{\tau_l,g}$)	0.999 ($\rho_{\tau_l,z}$, $\rho_{\tau_l,g}$)
ρ_{τ_x}	0.123	0.001	195.368	0.999987	0.986 ($\rho_{\tau_x,g}$)	0.998 ($\rho_{\tau_x,z}$, $\rho_{\tau_x,g}$)
ρ_{g,τ_x}	2.100	0.003	673.543	0.999999	0.992 (ρ_{z,τ_x})	0.999 ($\rho_{z,g}$, ρ_g)
$\rho_{z,g}$	4.853	0.004	1280.821	1.000000	0.992 (ρ_g)	0.999 (ρ_z, ρ_{z,τ_x})
$\rho_{\tau_l,g}$	1.396	0.006	229.345	0.999990	0.987 (ρ_{τ_l,τ_x})	0.999 ($\rho_{\tau_l,z}$, ρ_{τ_l,τ_x})
$\rho_{\tau_x,g}$	1.648	0.004	459.104	0.999998	0.986 (ρ_{τ_x})	0.999 ($\rho_{\tau_x,z}$, ρ_{τ_x})
ρ_g	0.107	0.000	1051.673	1.000000	0.992 ($\rho_{z,g}$)	0.999 ($\rho_{z,g}$, ρ_{g,τ_x})
q_{11}	0.554	0.036	15.585	0.997939	0.780 (q_{31})	0.788 (q_{21}, q_{31})
q_{21}	2.490	0.247	10.064	0.995051	0.747 (q_{41})	0.755 (q_{11}, q_{41})
q_{31}	1.555	0.038	40.815	0.999700	0.951 (q_{41})	0.960 (q_{11}, q_{41})
q_{41}	2.290	0.654	3.504	0.958405	0.951 (q_{31})	0.958 (q_{21}, q_{31})
q_{22}	0.716	0.045	15.827	0.998002	0.622 (q_{42})	0.632 ($\tau_{l_{ss}}$, q_{42})
q_{32}	10.978	0.388	28.313	0.999376	0.951 (q_{42})	0.951 ($\tau_{l_{ss}}$, q_{42})
q_{42}	1.894	0.062	30.769	0.999472	0.951 (q_{32})	0.955 (q_{22}, q_{32})
q_{33}	1.543	0.024	63.950	0.999878	0.909 (q_{43})	0.910 (ρ_{τ_x}, q_{43})
q_{43}	0.319	0.027	12.003	0.996524	0.909 (q_{33})	0.909 (z_{ss}, q_{33})
q_{44}	0.986	0.058	16.985	0.998265	0.113 (σ)	0.389 (δ, σ)
δ	0.129	0.013	9.575	0.994531	0.934 (g_{ss})	0.966 (z_{ss}, α)
σ	0.440	0.011	40.138	0.999690	0.940 (α)	0.980 ($\tau_{x_{ss}}$, ρ_{g,τ_l})
α	1.200	0.002	504.842	0.999998	0.940 (σ)	0.983 ($\tau_{x_{ss}}$, ρ_{g,τ_l})

B Appendix - Model Derivations

B.1 Representative Consumer

B.1.1 Optimization Problem of the Household

Suppose households own the capital stock and rent it out ar rate r_t. They also work for wages at rate w_t per unit of labor input and pay takes on labor, investment and bond holdings. Then, the optimization problem for the household looks as follows:

Objective function:

$$\max_{\{c_t(s^t), l_t(s^t), k_{t+1}(s^t), b_t(s^t)\}} \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t U\left(c_t(s^t), 1 - l(s^t)\right) \underbrace{N_t(s^t)}_{=(1+g_n)^t \text{ setting } N_0=1},$$

where $c_t(s^t),\ x_t(s^t) \geq 0,\ \forall s^t,\ t.$

Let $x_t(s^t)$ be any random variable. The *expectation* over the discounted sum of future possible realizations can be written as:

$$E_0 \sum_{t=0}^{\infty} \beta^t x_t(s^t) \equiv \sum_{t=0}^{\infty} \sum_{s^t} \beta^t x_t(s^t) \mu_t(s^t)$$

The second sum expresses that the expectation is a probability weighted average of the different possible realizations of the variable. We can thus rewrite the objective function as follows:

$$\max_{\{c_t(s^t), l_t(s^t), k_{t+1}(s^t), b_t(s^t)\}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t(s^t), 1 - l(s^t)\right) (1+g_n)^t,$$

Budget Constraint:

$$c_t(s^t) + [1 + \tau_x(s^t)] x_t(s^t) + [1 + \tau_b(s^t)] \left[(1+g_n) \frac{b_t(s^t)}{[1 + R_t(s^t)] p_t(s^t)} - \frac{b_{t-1}(s^{t-1})}{p_t(s^t)} \right]$$
$$= [1 - \tau_l(s^t)] w_t(s^t) l_t(s^t) + r_t(s^t) k_t(s^{t-1}) + T_t(s^t)$$

Capital Accumulation Law:

$$N_{t+1}(s^{t+1}) k_{t+1}(s^t) = [(1-\delta) k_t(s^t) + x_t(s^t)] N_t(s^t)$$
$$\iff (1+g_n) k_{t+1}(s^t) = (1-\delta) k_t(s^t) + x_t(s^t) \tag{B-1}$$

B.1.2 Lagrangian Function

Given that there is one budget constraint for each realization of s^t in each time period t one can write the Lagrangian function in the following way:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t(s^t), 1 - l(s^t)\right) N_t(s^t)$$
$$- \sum_{t=0}^{\infty} \sum_{s^t} \tilde{\lambda}_t \Big\{ c_t(s^t) + [1 + \tau_x(s^t)] x_t(s^t) + [1 + \tau_b(s^t)] \left[(1+g_n) \frac{b_t(s^t)}{[1 + R_t(s^t)] p_t(s^t)} - \frac{b_{t-1}(s^{t-1})}{p_t(s^t)} \right]$$
$$- [1 - \tau_l(s^t)] w_t(s^t) l_t(s^t) - r_t(s^t) k_t(s^{t-1}) - T_t(s^t) \Big\}$$

Making use of expression (B-1) for investment and recalling that the expectation is a probability weighted sum of the possible realizations, one can integrate the budget constraints into the first part of the function one obtains

$$
\begin{aligned}
\mathcal{L} = E_0 \Bigg\{ \sum_{t=0}^{\infty} \beta^t \Bigg[& U\left(c_t(s^t), 1 - l(s^t)\right) N_t(s^t) \\
& -\lambda_t \Bigg[c_t(s^t) + [1 + \tau_x(s^t)][(1+g_n)k_{t+1}(s^t) - (1-\delta)k_t(s^t)] \\
& + [1 + \tau_b(s^t)] \left[(1+g_n)\frac{b_t(s^t)}{[1 + R_t(s^t)]p_t(s^t)} - \frac{b_{t-1}(s^{t-1})}{p_t(s^t)} \right] \\
& - [1 - \tau_l(s^t)]w_t(s^t)l_t(s^t) - r_t(s^t)k_t(s^{t-1}) - T_t(s^t) \Bigg] \Bigg] \Bigg\}
\end{aligned}
$$

with $\lambda_t = \frac{\tilde{\lambda}_t}{\beta^t \mu_t(s^t)}$.

B.1.3 First Order Necessary Conditions

Differentiating the Lagrangian with respect to the choice variables of the household, one obtains the following first-order necessary conditions which hold $\forall s^t,\ t$:

For each state of the world, s^t, and each point of time t it holds that:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t(s^t)} &= \beta^t \mu_t(s^t)[U_{c,t}(s^t)(1+g_n)^t - \lambda_t] = 0 \\
&\iff \quad \lambda_t = U_{c,t}(s^t)(1+g_n)^t
\end{aligned}
\tag{B-2}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial n_t(s^t)} &= \beta^t \mu_t(s^t)[U_{l,t}(s^t)(1+g_n)^t + \lambda_t[1 - \tau_{l,t}(s^t)]w_t(s^t)] = 0 \\
&\iff \quad \lambda_t = -\frac{U_{l,t}(s^t)(1+g_n)^t}{[1 - \tau_{l,t}(s^t)]w_t(s^t)}
\end{aligned}
\tag{B-3}
$$

The *intratemporal* optimality condition is then given by

$$
-\frac{U_{l,t}(s^t)}{U_{c,t}(s^t)} = [1 - \tau_{l,t}(s^t)]w_t(s^t). \tag{B-4}
$$

When differentiating the Lagrangian with respect to capital $k_{t+1}(s^t)$, one has to note that at time $t+1$ the net capital stock of the previous period, $k_t(s^{t-1})$, becomes $k_{t+1}(s^t)$, which leads to an additional component in the derivative.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial k_{t+1}(s^t)} &= \beta^t \mu_t(s^t)\Big[-\lambda_t[1+\tau_{x,t}(s^t)](1+g_n)\Big] \\
&+ \sum_{s^{t+1}>s^t} \beta^{t+1}\mu_{t+1}(s^{t+1})\Big[\lambda_{t+1}[1+\tau_{x,t+1}(s^{t+1})](1-\delta)+r_{t+1}(s^{t+1})\Big] = 0 \\
\Longleftrightarrow 1 &= \frac{\beta}{(1+g_n)} \sum_{s^{t+1}>s^t} \frac{\mu_{t+1}(s^{t+1})}{\mu_t(s^t)} \frac{\lambda_{t+1}}{\lambda_t}\left[\frac{[1+\tau_{x,t+1}(s^{t+1})](1-\delta)+r_{t+1}(s^{t+1})}{1+\tau_{x,t}(s^t)}\right] \\
\Longleftrightarrow 1 &= \frac{\beta}{(1+g_n)} \sum_{s^{t+1}>s^t} \mu_{t+1}(s^{t+1}|s^t)\frac{U_{c,t+1}(s^{t+1})(1+g_n)^{t+1}}{U_{c,t}(s^t)(1+g_n)^t} \times \\
&\left[\frac{[1+\tau_{x,t+1}(s^{t+1})](1-\delta)+r_{t+1}(s^{t+1})}{1+\tau_{x,t}(s^t)}\right] \\
\Longleftrightarrow 1 &= \beta \sum_{s^{t+1}>s^t} \mu_{t+1}(s^{t+1}|s^t)\frac{U_{c,t+1}(s^{t+1})}{U_{c,t}(s^t)}\left[\frac{[1+\tau_{x,t+1}(s^{t+1})](1-\delta)+r_{t+1}(s^{t+1})}{1+\tau_{x,t}(s^t)}\right] \\
\Longleftrightarrow 1 &= \beta\, \mathbb{E}_t\left\{\frac{U_{c,t+1}(s^{t+1})}{U_{c,t}(s^t)}\left[\frac{[1+\tau_{x,t+1}(s^{t+1})](1-\delta)+r_{t+1}(s^{t+1})}{1+\tau_{x,t}(s^t)}\right]\right\},
\end{aligned}
\tag{B-5}
$$

where in the third step we used $\frac{\mu_{t+1}(s^{t+1})}{\mu_t(s^t)} \equiv \mu_{t+1}(s^{t+1}|s^t)$ and equation (B-2).

When differentiating the Lagrangian with respect to bonds, one has to note that at time $t+1$ the bonds of the previous period, $b_{t-1}(s^{t-1})$, become $b_t(s^t)$, which leads to an additional component in the derivative.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial b_t(s^t)} &= \beta^t \mu_t(s^t)\left[-\lambda_t[1+\tau_{b,t}(s^t)]\frac{(1+g_n)}{[1+R_t(s^t)]p_t(s^t)}\right] \\
&+ \sum_{s^{t+1}>s^t} \beta^{t+1}\mu_{t+1}(s^{t+1})\left[\lambda_{t+1}[1+\tau_{b,t+1}(s^{t+1})]\frac{1}{p_{t+1}(s^{t+1})}\right] = 0 \\
\Longleftrightarrow 1 &= \frac{\beta}{(1+g_n)} \sum_{s^{t+1}>s^t} \frac{\mu_{t+1}(s^{t+1})}{\mu_t(s^t)} \frac{\lambda_{t+1}}{\lambda_t} \frac{1+\tau_{b,t+1}(s^{t+1})}{1+\tau_{b,t}(s^t)} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})}[1+R_t(s^t)] \\
\Longleftrightarrow 1 &= \frac{\beta}{(1+g_n)} \sum_{s^{t+1}>s^t} \frac{\mu_{t+1}(s^{t+1})}{\mu_t(s^t)} \frac{U_{c,t+1}(s^{t+1})(1+g_n)^{t+1}}{U_{c,t}(s^t)(1+g_n)^t} \frac{1+\tau_{b,t+1}(s^{t+1})}{1+\tau_{b,t}(s^t)} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})}[1+R_t(s^t)] \\
\Longleftrightarrow 1 &= \beta \sum_{s^{t+1}>s^t} \mu_{t+1}(s^{t+1}|s^t)\frac{U_{c,t+1}(s^{t+1})}{U_{c,t}(s^t)} \frac{1+\tau_{b,t+1}(s^{t+1})}{1+\tau_{b,t}(s^t)} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})}[1+R_t(s^t)] \\
\Longleftrightarrow 1 &= \beta\, \mathbb{E}_t\left\{\frac{U_{c,t+1}(s^{t+1})}{U_{c,t}(s^t)} \frac{1+\tau_{b,t+1}(s^{t+1})}{1+\tau_{b,t}(s^t)} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})}[1+R_t(s^t)]\right\} \\
\Longleftrightarrow 1 &= \beta\, \mathbb{E}_t\left\{\frac{U_{c,t+1}(s^{t+1})}{U_{c,t}(s^t)} \frac{1+\tau_{b,t+1}(s^{t+1})}{1+\tau_{b,t}(s^t)} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})}[1+R_t(s^t)]\right\}
\end{aligned}
\tag{B-6}
$$

where in the fourth step we used $\frac{\mu_{t+1}(s^{t+1})}{\mu_t(s^t)} \equiv \mu_{t+1}(s^{t+1}|s^t)$ and equation (B-2).

B.2 Representative Producer

The representative producer operates an aggregate constant-returns-to-scale (CRS) production function

$$y_t(s^t) = F\left(k_t(s^{t-1}), Z_t(s^t)l_t(s^t)\right), \tag{B-7}$$

where $F(.,.)$ has the standard properties and $Z_t(s^t) = z_t(s^t)\left(1+g_z\right)^t$.

B.2.1 Optimization Problem of the Firm

The producer maximizes profits

$$y_t(s^t) - w_t(s^t)l_t(s^t) - r_t(s^t)k_t(s^{t-1}) \tag{B-8}$$

by setting

$$\begin{aligned} w_t(s^t) &= F_{l,t}\left(k_t(s^{t-1}), z_t(s^t)(1+g_z)^t l_t(s^t)\right) && \text{(B-9)} \\ r_t(s^t) &= F_{k,t}\left(k_t(s^{t-1}), z_t(s^t)(1+g_z)^t l_t(s^t)\right). && \text{(B-10)} \\ & && \text{(B-11)} \end{aligned}$$

B.3 Additional Model Equations

The aggregate resource constraint is given by

$$y_t(s^t) = c_t(s^t) + g_t(s^t) + x_t(s^t). \tag{B-12}$$

Monetary policy is assumed to set the interest rate according to a Taylor rule of the following type

$$R_t(s^t) = (1-\rho_R)\left[R + \omega_y(\ln\, y_t(s^t) - \ln\, y) + \omega_\pi(\pi_t(s^t) - \pi)\right] + \rho_R R_{t-1}(s^{t-1}) + \tilde{R}_t(s^t), \tag{B-13}$$

where $\rho_R \in [0,1)$, $\pi_t(s^t) \equiv \ln\, p_t(s^t) - \ln\, p_{t-1}(s^{t-1})$ is the inflation rate and a variable's symbol without a time subscript denotes the variable's steady-state (or balanced growth path) value. In addition, it is assumed that $\omega_\pi > 1$, thus eliminating explosive paths.

B.4 Functional Forms and Auxiliary Assumptions

From here on, we make the following functional form assumptions:

$F(k, Zl) = k^\alpha (Zl)^{1-\alpha}$ and $U(c, 1-l) = \frac{\left(c(1-l)^\psi\right)^{(1-\sigma)}}{(1-\sigma)}$

It is assumed that the state s_t follows a Markov process of the form $\mu(s^t|s^{t-1})$ and that the wedges in period t can be used to uncover the event s^t uniquely, in the sense that the mapping from the event s^t to the wedges $\left(z_t, 1-\tau_{l,t}, \frac{1}{1+\tau_{l,t}}, g_t, \frac{1}{1+\tau_{b,t}}, \tilde{R}_t\right)$ is one to one and onto. Given this assumption, without loss of generality, let the underlying event $s_t = (s_{zt}, s_{lt}, s_{xt}, s_{gt}, s_{bt}, s_{\tilde{R}t})$, and let $\log\, z_t(s^t) = s_{zt}$, $\tau_{l,t}(s^t) = s_{lt}$, $\tau_{x,t}(s^t) = s_{xt}$, and $\log\, g_t(s^t) = s_{gt}$. Given the unique mapping between s_t and the wedges we make following auxiliary choices:

$$\log\, z_t = \log\, z(s^t),\ \log\, \hat{g}_t = \log\, \hat{g}_t(s^t),\ \tau_{l,t} = \tau_l(s^t),\ \tau_{x,t} = \tau_x(s^t),\ \tau_{b,t} = \tau_b(s^t),\ \tilde{R}_t = \tilde{R}(s^t)$$

Note that we have effectively assumed that agents use only past wedges to forecast future wedges and that the wedges in period t are sufficient statistics for the event in period t. More precisely, the VAR representation of the underlying state s_t is modeled as follows

$$s_{t+1} = P_0 + P s_t + Q\varepsilon_{s,t+1},$$

where $\varepsilon_{s,t+1} \sim N(0, I)$.

B.5 Operational Model

In the operational model we consider quantities which are not only expressed in per-capita terms but also detrended. To highlight the differences between this model's and the previous model's variables we introduce the notation $\left(\hat{v} \equiv \frac{v_t}{(1+g_z)^t} \equiv \frac{V_t}{N_t(1+g_z)^t}\right)$. The model is then given by:

- CRS Production Function

$$\hat{y}_t(s^t) = \hat{k}_t(s^{t-1})^{\alpha} \left(z_t l_t(s^t)\right)^{1-\alpha} \tag{B-14}$$

- Aggregate Resource Constraint

$$\hat{y}_t(s^t) = \hat{c}_t(s^t) + \hat{g}_t + \hat{x}_t(s^t) \tag{B-15}$$

- Capital Accumulation Law

$$\begin{aligned} (1+g_n)(1+g_z)^{t+1}\hat{k}_{t+1}(z^t) &= (1-\delta)(1+g_z)^t\hat{k}_t(z^{t-1}) + (1+g_z)^t\hat{x}_t(z^t) \\ \Longleftrightarrow (1+g_n)(1+g_z)\hat{k}_{t+1}(z^t) &= (1-\delta)\hat{k}_t(z^{t-1}) + \hat{x}_t(z^t) \end{aligned} \tag{B-16}$$

- Taylor Rule

$$R_t(s^t) = (1-\rho_R)\left[R + \omega_y(\ln\, \hat{y}_t(s^t) - \ln\, \hat{y}) + \omega_\pi(\pi_t(s^t) - \pi)\right] + \rho_R R_{t-1}(s^{t-1}) + \tilde{R}_t, \tag{B-17}$$

- F.O.C. Labor

$$\psi\frac{\hat{c}_t(s^t)}{1-l_t(s^t)} = (1-\tau_{l,t})(1-\alpha)\hat{k}_t(s^{t-1})^{\alpha} z_t^{1-\alpha} l_t(s^t)^{-\alpha} \tag{B-18}$$

- F.O.C. Capital

$$\begin{aligned} 1 &= \beta\mathbb{E}_t\Bigg\{\frac{\left(\hat{c}_{t+1}(1+g_z)^{t+1}\right)^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}}{\left(\hat{c}_t(1+g_z)^t\right)^{-\sigma}(1-l_t)^{\psi(1-\sigma)}}\times \\ &\quad \left[\frac{(1+\tau_{x,t+1})(1-\delta) + \alpha\hat{k}_{t+1}(s^t)^{\alpha-1}\left(z_{t+1}l_{t+1}(s^{t+1})\right)^{1-\alpha}}{1+\tau_{x,t}}\right]\Bigg\} \\ &= \tilde{\beta}\mathbb{E}_t\Bigg\{\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{-\sigma}\left(\frac{1-l_{t+1}}{1-l_t}\right)^{\psi(1-\sigma)}\left[\frac{(1+\tau_{x,t+1})(1-\delta) + \alpha\hat{k}_{t+1}(s^t)^{\alpha-1}\left(z_{t+1}l_{t+1}(s^{t+1})\right)^{1-\alpha}}{1+\tau_{x,t}}\right]\Bigg\}, \end{aligned} \tag{B-19}$$

  where $\tilde{\beta} = \beta/(1+g_z)^{-\sigma}$.

- F.O.C. Bonds

$$\begin{aligned} 1 &= \beta \mathbb{E}_t \Big\{ \frac{\hat{c}_t(s^t)(1+g_z)^t}{\hat{c}_{t+1}(s^{t+1})(1+g_z)^{t+1}} \frac{1+\tau_{b,t+1}}{1+\tau_{b,t}} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})} [1+R_t(s^t)] \Big\} \\ \Longleftrightarrow 1 &= \tilde{\beta} \mathbb{E}_t \Big\{ \frac{\hat{c}_t(s^t)}{\hat{c}_{t+1}(s^{t+1})} \frac{1+\tau_{b,t+1}}{1+\tau_{b,t}} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})} [1+R_t(s^t)] \Big\} \end{aligned} \tag{B-20}$$

where $\tilde{\beta} = \beta/(1+g_z)$.

B.6 Steady State

The model in steady-state is given by:

- CRS Production Function

$$\hat{y} = \hat{k}^{\alpha} (zl)^{1-\alpha} \tag{B-21}$$

- Aggregate Resource Constraint

$$\hat{y} = \hat{c} + \hat{g} + \hat{x} \tag{B-22}$$

- Capital Accumulation Law

$$(1+g_n)(1+g_z)\hat{k} = (1-\delta)\hat{k} + \hat{x} \tag{B-23}$$

- Taylor Rule

$$\pi = \frac{-\tilde{R}}{(1-\rho_R)\omega_\pi} + \pi_{SS}, \tag{B-24}$$

- F.O.C. Labor

$$\psi \frac{\hat{c}}{1-l} = (1-\tau_l)(1-\alpha)\hat{k}^{\alpha} z^{1-\alpha} l^{-\alpha} \tag{B-25}$$

- F.O.C. Capital

$$1 = \tilde{\beta} \frac{(1+\tau_x)(1-\delta) + \alpha \hat{k}^{\alpha-1} (zl)^{1-\alpha}}{1+\tau_x} \tag{B-26}$$

- F.O.C. Bonds

$$1 = \tilde{\beta}\exp(-\pi)(1+R) \Rightarrow R = \frac{1}{\tilde{\beta}\ \exp(-\pi)} - 1, \tag{B-27}$$

where we used that $\pi_t = \log\left(\frac{p_t}{p_{t-1}}\right)$.

To solve for the steady state of real variables, we start by solving (B-26) w.r.t. $\hat{k}$:

$$\hat{k} = \left[\frac{\alpha \tilde{\beta}}{(1+\tau_x)[1-\tilde{\beta}(1-\delta)]} \right]^{\frac{1}{1-\alpha}} zl \equiv \Lambda zl \tag{B-28}$$

Plugging this expression for $\hat{k}$ in equation (B-25) yields:

$$\begin{aligned} \psi\frac{\hat{c}}{1-l} &= (1-\tau_l)(1-\alpha)(\Lambda zl)^{\alpha}z^{1-\alpha}l^{-\alpha} \\ \Leftrightarrow \psi\frac{\hat{c}}{1-l} &= (1-\tau_l)(1-\alpha)\Lambda^{\alpha}z \\ \Leftrightarrow \hat{c} &= \frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}z(1-l) \end{aligned} \tag{B-29}$$

Inserting (B-28), (B-29) and (B-23) into the aggregate resource constraint leads to

$$\begin{aligned} \hat{k}^{\alpha}(zl)^{1-\alpha} &= \hat{c}+\hat{g}+(1+g_n)(1+g_z)\hat{k}-(1-\delta)\hat{k} \\ (\Lambda zl)^{\alpha}(zl)^{1-\alpha} &= \frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}z(1-l)+\hat{g} \qquad \text{(B-30)} \\ &\quad +\left[(1+g_n)(1+g_z)-(1-\delta)\right]\Lambda zl \\ \Lambda^{\alpha}zl &= \frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}z-\frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}zl+\hat{g} \\ &\quad +\left[(1+g_n)(1+g_z)-(1-\delta)\right]\Lambda zl \end{aligned}$$

$$\begin{aligned} &\Lambda^{\alpha}zl+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}zl \\ &-\left[(1+g_n)(1+g_z)-(1-\delta)\right]\Lambda zl = \hat{g}+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}z \\ &l = \frac{\hat{g}+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}z}{\Lambda^{\alpha}z\left[1+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\right]-\Lambda z\left[(1+g_n)(1+g_z)-(1-\delta)\right]} \end{aligned} \tag{B-31}$$

Using (B-31) in (B-28) one obtains:

$$\begin{aligned} \hat{k}=\Lambda zl &= \Lambda z\frac{\hat{g}+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}z}{\Lambda^{\alpha}z\left[1+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\right]-\Lambda z\left[(1+g_n)(1+g_z)-(1-\delta)\right]} \\ &= \frac{\hat{g}+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\Lambda^{\alpha}z}{\Lambda^{\alpha-1}\left[1+\frac{1}{\psi}(1-\tau_l)(1-\alpha)\right]-\left[(1+g_n)(1+g_z)-(1-\delta)\right]} \end{aligned} \tag{B-32}$$

B.7 Definitions

Below we define the notation used for the model's variables and parameters.

B.7.1 Variables

Lower-case variables define quantities in per-capita terms as follows:

Z: labor-augmenting technical change ($Z = z(1 + g_z)^t$)
b: one-period, nominal riskfree bonds; purchased in period t, pay off in period $t + 1$
c: consumption
g: government consumption
k: net capital stock
l: labor
N: population ($N_t = N_0(1 + g_n)^t$)
r: rental rate on capital
R: nominal interest rate
t: time period
w: wage rate
x: investment
y: output
s: state of the world at time t
π: inflation rate
τ_l: labor tax
τ_x: tax on investment
τ_b: tax on bond holdings

B.7.2 Parameters

The following parameters appear in the model:

g_n: population growth rate of labor-augmenting technological process
g_z: growth rate of labor-augmenting technological process
α: parameter which determines the share of (and weight on) net capital stock in the Cobb-Douglas CRS production function
β: subjective discount factor, reflecting the time preference of the household
δ: depreciation rate of net capital stock
ψ: Frisch elasticity of labor supply
ρ_R: weight on lagges nominal interest rate in Taylor rule (extent of "interest rate smoothing")
ω_π: coefficient on deviations of inflation from its steady state value in Taylor rule
ω_y: coefficient on deviations of output from its steady state value in Taylor rule

C Appendix - Gensys State Space

C.1 Log-Linearized Equilibrium Conditions

We start by writing the system of equations in terms of k and s. This is done by replacing r, w, $\hat{c}$, and $\hat{x}$ in the first-order conditions with functions of the states. Thus we start with

$$\hat{c}_t + \hat{g}_t + (1+g_z)(1+g_n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t = \hat{y}_t = \hat{k}_t^{\alpha}(z_t l_t)^{1-\alpha} \tag{C-1}$$

$$\frac{\psi \hat{c}_t}{1-l_t} = (1-\tau_{lt})(1-\alpha)\hat{k}_t^{\alpha} l_t^{-\alpha} z_t^{1-\alpha} \tag{C-2}$$

$$\begin{aligned}
&(1+\tau_{xt})\hat{c}_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\
&\quad = \hat{\beta} E_t \hat{c}_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}[\alpha \hat{k}_{t+1}^{\alpha-1}(z_{t+1}l_{t+1})^{1-\alpha} + (1-\delta)(1+\tau_{xt+1})], \\
&,
\end{aligned} \tag{C-3}$$

where $\psi = \lambda/(1-\lambda)$ and which can be reduced to the following:

$$\begin{aligned}
&\psi[\hat{k}_t^{\alpha}(z_t l_t)^{1-\alpha} - (1+g_n)(1+g_z)\hat{k}_{t+1} + (1-\delta)\hat{k}_t - \hat{g}_t] \\
&\quad = (1-\tau_{lt})(1-\alpha)\hat{k}_t^{\alpha} l_t^{-\alpha} z_t^{1-\alpha}(1-l_t) \\
&(1+\tau_{xt})[\hat{k}_t^{\alpha}(z_t l_t)^{1-\alpha} - (1+g_n)(1+g_z)\hat{k}_{t+1} + (1-\delta)\hat{k}_t - \hat{g}_t]^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\
&\quad = \hat{\beta} E_t[\hat{k}_{t+1}^{\alpha}(z_{t+1}l_{t+1})^{1-\alpha} - (1+g_n)(1+g_z)\hat{k}_{t+2} \\
&\qquad + (1-\delta)\hat{k}_{t+1} - \hat{g}_{t+1}]^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)} \\
&\qquad [\alpha \hat{k}_{t+1}^{\alpha-1}(z_{t+1}l_{t+1})^{1-\alpha} + (1-\delta)(1+\tau_{xt+1})].
\end{aligned}$$

Next, we compute the steady state of the system for constant values for z, the taxes, and government spending:

$$\begin{aligned}
\hat{k}/l &= \left(\frac{(1+\tau_x)(1-\hat{\beta}(1-\delta))}{\hat{\beta}\alpha z^{1-\alpha}} \right)^{1/(\alpha-1)} \\
\hat{c} &= \left[(\hat{k}/l)^{\alpha-1} z^{1-\alpha} - (1+g_z)(1+g_n) + 1 - \delta \right] \hat{k} - \hat{g} = \xi_1 \hat{k} - \hat{g} \\
\hat{c} &= \left[(1-\tau_l)(1-\alpha)(\hat{k}/l)^{\alpha} z^{1-\alpha}/\psi \right] (1 - 1/(\hat{k}/l)\ \hat{k}) = \xi_2 - \xi_3 \hat{k},
\end{aligned}$$

where the last two equations imply $\hat{k} = (\xi_2 + \hat{g})/(\xi_1 + \xi_3)$, $\hat{c} = \xi_1 \hat{k} - \hat{g}$, $l = (1/(\hat{k}/l))\hat{k}$.

The log-linearization is done around these steady-state values. Detrended consumption is obtained via (C-1) and given approximately by

$$\begin{aligned}
\hat{c}_t &\approx \hat{c} \log \hat{c}_t \\
&\approx \hat{k}^{\alpha}(zl)^{1-\alpha}[\alpha \log \hat{k}_t + (1-\alpha)(\log z_t + \log l_t)] \\
&\quad - (1+g_z)(1+g_n)\hat{k} \log \hat{k}_{t+1} + (1-\delta)\hat{k} \log \hat{k}_t - \hat{g} \log \hat{g}_t.
\end{aligned}$$

The labor input is then derived from the static first-order condition (C-2):

$$\begin{aligned}
0 \approx \psi\big\{ &\hat{k}^{\alpha}(zl)^{1-\alpha}[\alpha \log \hat{k}_t + (1-\alpha)(\log z_t + \log l_t)] \\
&- (1+g_z)(1+g_n)\hat{k} \log \hat{k}_{t+1} + (1-\delta)\hat{k} \log \hat{k}_t - \hat{g} \log \hat{g}_t \big\} \\
&\quad + (1-\alpha)(1-\tau_l)\hat{k}^{\alpha} l^{-\alpha} z^{1-\alpha}(1-l)\big\{1/(1-\tau_l)\, \tau_{lt} \\
&\qquad - \alpha \log \hat{k}_t + \alpha \log l_t - (1-\alpha)\log z_t + l/(1-l) \log l_t \big\},
\end{aligned}$$

which we write succinctly as

$$\log l_t = \phi_{lk} \log \hat{k}_t + \phi_{lz} \log z_t + \phi_{ll}\tau_{lt} + \phi_{lg} \log \hat{g}_t + \phi_{lk'} \log \hat{k}_{t+1}.$$

Using this equation for $\log l$, we use the other static first-order conditions to write $\log \hat{y}$, $\log \hat{x}$, and $\log \hat{c}$ as follows:

$$\begin{aligned}
\log \hat{y}_t &= \phi_{yk} \log \hat{k}_t + \phi_{yz} \log z_t + \phi_{yl}\tau_{lt} + \phi_{yg} \log \hat{g}_t + \phi_{yk'} \log \hat{k}_{t+1} \\
&= (\alpha + (1-\alpha)\phi_{lk}) \log \hat{k}_t + (1-\alpha)(1+\phi_{lz}) \log z_t \\
&\qquad + (1-\alpha)[\phi_{ll}\tau_{lt} + \phi_{lk'} \log \hat{k}_{t+1}] \\
\log \hat{x}_t &= (1+g_z)(1+g_n)\hat{k}/\hat{x} \log \hat{k}_{t+1} - (1-\delta)\hat{k}/\hat{x} \log \hat{k}_t \\
\log \hat{c}_t &= \phi_{ck} \log \hat{k}_t + \phi_{cz} \log z_t + \phi_{cl}\tau_{lt} + \phi_{cg} \log \hat{g}_t + \phi_{ck'} \log \hat{k}_{t+1} \\
&= [\hat{y} \log y_t - \hat{x} \log x_t - \hat{g} \log \hat{g}_t]/\hat{c},
\end{aligned}$$

where the ϕ's are known functions of the parameters.

Capital is derived from the dynamic first-order condition (C-3)

$$\begin{aligned}
0 \approx{} & (1+\tau_x)\hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)} \{-\psi(1-\sigma)l/(1-l) \log l_t - \sigma \log \hat{c}_t\} \\
& + \hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)}\tau_{xt} \\
& - \hat{\beta} E_t\big\{\big[\alpha \hat{k}^{\alpha-1}(zl)^{1-\alpha} + (1-\delta)(1+\tau_x)\big] \\
& \quad \cdot \big[\hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)} \{-\psi(1-\sigma)l/(1-l) \log l_{t+1} - \sigma \log \hat{c}_{t+1}\}\big] \\
& \quad + \hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)}\big[\alpha \hat{k}^{\alpha-1}(zl)^{1-\alpha}(1-\alpha) \\
& \qquad \cdot (\log l_{t+1} + \log z_{t+1} - \log \hat{k}_{t+1}) + (1-\delta)\tau_{xt+1}\big]\big\},
\end{aligned}$$

which simplifies to

$$\begin{aligned}
0 \approx{} & (1+\tau_x)\{-\psi(1-\sigma)l/(1-l) \log l_t - \sigma \log \hat{c}_t\} + \tau_{xt} \\
& - E_t\big\{(1+\tau_x)\{-\psi(1-\sigma)l/(1-l) \log l_{t+1} - \sigma \log \hat{c}_{t+1}\} \\
& \quad + \hat{\beta}\big[r(1-\alpha)(\log l_{t+1} + \log z_{t+1} - \log \hat{k}_{t+1}) + (1-\delta)\tau_{xt+1}\big]\big\},
\end{aligned}$$

where $r = \alpha\hat{y}/\hat{k}$ and can be rewritten as

$$\begin{aligned}
0 \approx{} & \phi_{kl} \log l_t + \phi_{kc} \log \hat{c}_t + \tau_{xt} + \phi_{k\mathbb{E}l}\mathbb{E}_t \log l_{t+1} + \phi_{k\mathbb{E}c}\mathbb{E}_t \log \hat{c}_{t+1} \\
& + \phi_{k\mathbb{E}z}\mathbb{E}_t \log z_{t+1} + \phi_{k\mathbb{E}k}\mathbb{E}_t \log \hat{k}_{t+1} + \phi_{k\mathbb{E}tx}\mathbb{E}_t\tau_{xt+1}. \qquad \text{(C-4)}
\end{aligned}$$

C.2 Allowing for Adjustment Costs

To do log-linear computation (as in the baseline economy) in the case with adjustment costs and $\tau_{ct} = \tau_{kt} = 0$, we start with

$$\hat{c}_t + \hat{g}_t + (1+g_z)(1+g_n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t + \varphi(\hat{x}_t/\hat{k}_t)\hat{k}_t = \hat{y}_t = \hat{k}_t^{\alpha}(z_t l_t)^{1-\alpha}$$

$$\frac{\psi \hat{c}_t}{1-l_t} = (1-\tau_{lt})(1-\alpha)\hat{k}_t^{\alpha} l_t^{-\alpha} z_t^{1-\alpha}$$

$$\begin{aligned}(1+\tau_{xt})\hat{c}_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)}/\big(1-\varphi'(\hat{x}_t/\hat{k}_t)\big) \\ = \hat{\beta} E_t \hat{c}_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}\Big[\alpha \hat{k}_{t+1}^{\alpha-1}(z_{t+1}l_{t+1})^{1-\alpha} + \big(1-\delta \\ - \varphi(\hat{x}_{t+1}/\hat{k}_{t+1}) + \varphi'(\hat{x}_{t+1}/\hat{k}_{t+1})\hat{x}_{t+1}/\hat{k}_{t+1}\big) \\ (1+\tau_{xt+1})/\Big(1-\varphi'(\hat{x}_{t+1}/\hat{k}_{t+1})\Big)\Big],\end{aligned}$$

where

$$\varphi(x/k) = \frac{a}{2}\left(\frac{x}{k} - b\right)^2 .$$

and b is set equal to the investment-capital trend rate (i.e., $b = (1+g_z)(1+g_n) - 1 + \delta$). To allow for different intensities of adjustment costs, a is raised from 0 (no adjustment costs) to 12.88 (the level used by Bernanke et al. (1998), the normal adjustment costs BGG level) to 4*12.88 (extreme adjustment costs).

Assuming $\varphi(\hat{x}/\hat{k}) = \varphi'(\hat{x}/\hat{k}) = 0$, the log-linearization of these equations yields the same results as in the benchmark with the exception of the intertemporal condition:

$$\begin{aligned}0 \approx (1+\tau_x)\big\{-\psi(1-\sigma)l/(1-l)\log l_t - \sigma \log \hat{c}_t + \eta(\log \hat{x}_t - \log \hat{k}_t)\big\} + \tau_{xt} \\ - E_t\Big\{(1+\tau_x)\left\{-\psi(1-\sigma)l/(1-l)\log l_{t+1} - \sigma\log \hat{c}_{t+1}\right\} \\ + \hat{\beta}\big[r(1-\alpha)(\log l_{t+1} + \log z_{t+1} - \log \hat{k}_{t+1}) \\ + (1+\tau_x)(1+g_z)(1+g_n)\eta(\log \hat{x}_{t+1} - \log \hat{k}_{t+1}) \\ + (1-\delta)\tau_{xt+1}\big]\Big\},\end{aligned}$$

where $r = \alpha\hat{y}/\hat{k}$, $\eta = \varphi''(\hat{x}/\hat{k})(\hat{x}/\hat{k}) = ab$ and the term in red is what is added to the baseline log-linearized dynamic equilibrium condition due to the presence of adjustment costs. This system can be rewritten as

$$\begin{aligned}0 \approx \phi_{kl}\log l_t + \phi_{kc}\log \hat{c}_t + \tau_{xt} + \phi_{kx}\log \hat{x}_t + \phi_{kk}\log \hat{k}_t + \phi_{k\mathbb{E}l}\mathbb{E}_t \log l_{t+1} + \phi_{k\mathbb{E}c}\mathbb{E}_t\log \hat{c}_{t+1} \\ + \phi_{k\mathbb{E}z}\mathbb{E}_t\log z_{t+1} + \phi^{adj}_{k\mathbb{E}k}\mathbb{E}_t\log \hat{k}_{t+1} + \phi_{k\mathbb{E}tx}\mathbb{E}_t\tau_{xt+1} + \phi^{adj}_{k\mathbb{E}x}\mathbb{E}_t x_{t+1}.\end{aligned} \quad \text{(C-5)}$$

and differs from its baseline counterpart (C-4) by the terms in red.

C.3 Extension to Monetary BCA - Šustek (2011)

The Monetary BCA model features also bonds and prices. We thus have two additional equations which describe the dynamics of the nominal interest rate and inflation. The Taylor Rule takes the form

$$\begin{aligned} R_t &= (1-\rho_R)\left[R + \omega_y(\log \hat{y}_t - \log \hat{y}) + \omega_\pi(\pi_t - \pi)\right] + \rho_R R_{t-1} + \tilde{R}_t \\ \Leftrightarrow R_t &= \phi_{\pi 0} + \phi_{\pi y} \log \hat{y}_t(s^t) + \phi_\pi \pi_t + \phi_{\pi R} R_{t-1} + \tilde{R}_t. \end{aligned} \tag{C-6}$$

whereas the first order condition for bonds is given by

$$(1+\tau_{bt})\hat{c}_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} = \hat{\beta} E_t \hat{c}_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}[(1+\tau_{bt+1})\exp(-\pi_{t+1})(1+R_t)], \tag{C-7}$$

We follow the lines of CKM (2007) for the log-linearization of the F.O.C. for bonds and obtain

$$\begin{aligned} 0 \approx\ & (1+\tau_b)\hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)}\left\{-\psi(1-\sigma)l/(1-l)\log l_t - \sigma \log \hat{c}_t\right\} \\ & \quad + \hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)}\tau_{bt} \\ & - \hat{\beta} E_t \Big\{ \big[(1+\tau_b)\exp(-\pi)(1+R)\big] \\ & \quad \cdot \big[\hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)}\left\{-\psi(1-\sigma)l/(1-l)\log l_{t+1} - \sigma \log \hat{c}_{t+1}\right\}\big] \\ & \quad + \hat{c}^{-\sigma}(1-l)^{\psi(1-\sigma)}\big[(1+\tau_b)\exp(-\pi)(1+R) \\ & \qquad \cdot \big((1+\tau_b)^{-1}\tau_{bt+1} - \pi_{t+1} + R_t\big)\big]\Big\}, \end{aligned}$$

which simplifies to

$$\begin{aligned} 0 \approx\ & \left\{-\psi(1-\sigma)l/(1-l)\log l_t - \sigma \log \hat{c}_t\right\} + (1+\tau_b)^{-1}\tau_{bt} \\ & - E_t\Big\{\big[-\psi(1-\sigma)l/(1-l)\log l_{t+1} - \sigma \log \hat{c}_{t+1}\big] \\ & \quad + \big[(1+\tau_b)^{-1}\tau_{bt+1} - \pi_{t+1} + R_t\big]\Big\}, \end{aligned}$$

and can be rewritten as

$$0 \approx \phi_{Rl}\log l_t + \phi_{Rc}\log \hat{c}_t + \phi_{R\tau_b}\tau_{bt} + \phi_{R\mathbb{E}l}\mathbb{E}_t \log l_{t+1} + \phi_{R\mathbb{E}c}\mathbb{E}_t \log \hat{c}_{t+1} + \phi_{R\mathbb{E}\tau_b}\mathbb{E}_t \tau_{bt+1} + \mathbb{E}_t \pi_{t+1} - R_t.$$

C.4 Gensys State Space Representation

C.4.1 BCA - Chari et al. (2007)

The models we are interested in can be cast in the form

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + C + \Psi \epsilon_t + \Pi \eta_t \tag{C-8}$$

$t = 1, \ldots, T$, where C is a vector of constants, ϵ_t is an exogenously evolving, possibly serially correlated, random disturbance, and η_t is an expectational error, satisfying $E_t \eta_{t+1} = 0$, all t. The η_t terms are not given exogenously, but instead are treated as determined as part of the model solution.

Within the context of the BCA model, the matrices $\Gamma_0 y_t$, $\Gamma_1 y_{t-1}$, C, $\Psi \epsilon_t$, $\Pi \eta_t$ are given by

$$\Gamma_0\, y_t = \begin{bmatrix}
\phi_{k\mathbb{E}k} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \phi_{kl} & 0 & \phi_{kc} & \phi_{k\mathbb{E}l} & \phi_{k\mathbb{E}c} & \phi_{k\mathbb{E}z} & \phi_{k\mathbb{E}\tau_x} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{yk'} & \phi_{yz} & \phi_{y\tau_l} & 0 & \phi_{yg} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{xk'} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{lk'} & \phi_{lz} & \phi_{l\tau_l} & 0 & \phi_{lg} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \phi_{cg} & 0 & \phi_{cy} & \phi_{cx} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\log \hat{k}_{t+1} \\ \log z_t \\ \tau_{lt} \\ \tau_{xt} \\ \log \hat{g}_t \\ 1 \\ \log \hat{y}_t \\ \log \hat{x}_t \\ \log l_t \\ \log \hat{g}_t^{obs} \\ \log \hat{c}_t \\ \mathbb{E}_t\{\log l_{t+1}\} \\ \mathbb{E}_t\{\log \hat{c}_{t+1}\} \\ \mathbb{E}_t\{\log z_{t+1}\} \\ \mathbb{E}_t\{\tau_{x,t+1}\}
\end{bmatrix}$$

$$\Gamma_1\, y_{t-1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & \rho_{z,\tau_l} & \rho_{z,\tau_x} & \rho_{z,g} & \bar{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_l,z} & \rho_{\tau_l} & \rho_{\tau_l,\tau_x} & \rho_{\tau_l,g} & \bar{\tau}_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_x,z} & \rho_{\tau_x,\tau_l} & \rho_{\tau_x} & \rho_{\tau_x,g} & \bar{\tau}_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{g,z} & \rho_{g,\tau_l} & \rho_{g,\tau_x} & \rho_g & \bar{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{yk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{xk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{lk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\log \hat{k}_t \\ \log z_{t-1} \\ \tau_{lt-1} \\ \tau_{xt-1} \\ \log \hat{g}_{t-1} \\ 1 \\ \log \hat{y}_{t-1} \\ \log \hat{x}_{t-1} \\ \log l_{t-1} \\ \log \hat{g}_{t-1}^{obs} \\ \log \hat{c}_{t-1} \\ \mathbb{E}_{t-1}\{\log l_t\} \\ \mathbb{E}_{t-1}\{\log \hat{c}_t\} \\ \mathbb{E}_{t-1}\{\log z_t\} \\ \mathbb{E}_{t-1}\{\tau_{x,t}\}
\end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

$$\Psi\ \epsilon_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q11 & 0 & 0 & 0 \\ q21 & q22 & 0 & 0 \\ q31 & q32 & q33 & 0 \\ q41 & q42 & q43 & q44 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{z,t} \\ \varepsilon_{\tau_l,t} \\ \varepsilon_{\tau_x,t} \\ \varepsilon_{g,t} \end{bmatrix}$$

$$\Pi\ \eta_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{E}l,t} \\ \eta_{\mathbb{E}c,t} \\ \eta_{\mathbb{E}z,t} \\ \eta_{\mathbb{E}\tau_x,t} \end{bmatrix}$$

C.4.2 Monetary BCA - Šustek (2011)

Within the context of the MBCA model, the matrices $\Gamma_0 y_t$, $\Gamma_1 y_{t-1}$, C, $\Psi\epsilon_t$, $\Pi\eta_t$ are given by

$$\Gamma_0\, y_t = \begin{bmatrix}
\phi_{k\mathbb{E}k} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{kl} & 0 & 0 & 0 & \phi_{kc} & \phi_{k\mathbb{E}l} & \phi_{k\mathbb{E}c} & \phi_{k\mathbb{E}z} & \phi_{k\mathbb{E}\tau_x} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{yk'} & \phi_{yz} & \phi_{y\tau_l} & 0 & \phi_{yg} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{xk'} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{lk'} & \phi_{lz} & \phi_{l\tau_l} & 0 & \phi_{lg} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \phi_{\pi_0} & \phi_{\pi y} & 0 & 0 & 0 & \phi_{\pi} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \phi_{R\tau_b} & 0 & 0 & 0 & 0 & \phi_{Rl} & 0 & 0 & -1 & \phi_{Rc} & \phi_{R\mathbb{E}l} & \phi_{R\mathbb{E}c} & 0 & 0 & \phi_{R\mathbb{E}\tau_b} & 1 \\
0 & 0 & 0 & 0 & \phi_{cg} & 0 & 0 & 0 & \phi_{cy} & \phi_{cx} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\log \hat{k}_{t+1} \\ \log z_t \\ \tau_{lt} \\ \tau_{xt} \\ \log \hat{g}_t \\ \tau_{bt} \\ \tilde{R}_t \\ 1 \\ \log \hat{y}_t \\ \log \hat{x}_t \\ \log l_t \\ \log \hat{g}_t^{obs} \\ \pi_t \\ R_t \\ \log \hat{c}_t \\ \mathbb{E}_t\{\log l_{t+1}\} \\ \mathbb{E}_t\{\log \hat{c}_{t+1}\} \\ \mathbb{E}_t\{\log z_{t+1}\} \\ \mathbb{E}_t\{\tau_{x,t+1}\} \\ \mathbb{E}_t\{\tau_{b,t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\}
\end{bmatrix}$$

$$
\Gamma_1 \; y_{t-1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & \rho_{z,\tau_l} & \rho_{z,\tau_x} & \rho_{z,g} & \rho_{z,\tau_b} & \rho_{z,\tilde{R}} & \bar{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_l,z} & \rho_{\tau_l} & \rho_{\tau_l,\tau_x} & \rho_{\tau_l,g} & \rho_{\tau_l,\tau_b} & \rho_{\tau_l,\tilde{R}} & \bar{\tau}_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_x,z} & \rho_{\tau_x,\tau_l} & \rho_{\tau_x} & \rho_{\tau_x,g} & \rho_{\tau_x,\tau_b} & \rho_{\tau_x,\tilde{R}} & \bar{\tau}_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{g,z} & \rho_{g,\tau_l} & \rho_{g,\tau_x} & \rho_g & \rho_{g,\tau_b} & \rho_{g,\tilde{R}} & \bar{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_b,z} & \rho_{\tau_b,\tau_l} & \rho_{\tau_b,\tau_x} & \rho_{\tau_b,g} & \rho_{\tau_b} & \rho_{\tau_b,\tilde{R}} & \bar{\tau}_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tilde{R},z} & \rho_{\tilde{R},\tau_l} & \rho_{\tilde{R},\tau_x} & \rho_{\tilde{R},g} & \rho_{\tilde{R},\tau_b} & \rho_{\tilde{R}} & \bar{\tilde{R}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{yk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{xk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{lk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\phi_{\pi R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\log \hat{k}_t \\
\log z_{t-1} \\
\tau_{lt-1} \\
\tau_{xt-1} \\
\log \hat{g}_{t-1} \\
\tau_{bt-1} \\
\tilde{R}_{t-1} \\
1 \\
\log \hat{y}_{t-1} \\
\log \hat{x}_{t-1} \\
\log l_{t-1} \\
\log \hat{g}^{obs}_{t-1} \\
\pi_{t-1} \\
R_{t-1} \\
\log \hat{c}_{t-1} \\
\mathbb{E}_{t-1}\{\log l_t\} \\
\mathbb{E}_{t-1}\{\log \hat{c}_t\} \\
\mathbb{E}_{t-1}\{\log z_t\} \\
\mathbb{E}_{t-1}\{\tau_{x,t}\} \\
\mathbb{E}_{t-1}\{\tau_{b,t}\} \\
\mathbb{E}_{t-1}\{\pi_t\}
\end{bmatrix}
$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

$$\Psi\ \epsilon_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ q11 & 0 & 0 & 0 & 0 & 0 \\ q21 & q22 & 0 & 0 & 0 & 0 \\ q31 & q32 & q33 & 0 & 0 & 0 \\ q41 & q42 & q43 & q44 & 0 & 0 \\ q51 & q52 & q53 & q54 & q55 & 0 \\ q61 & q62 & q63 & q64 & q65 & q66 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{z,t} \\ \varepsilon_{\tau_l,t} \\ \varepsilon_{\tau_x,t} \\ \varepsilon_{g,t} \\ \varepsilon_{\tau_b,t} \\ \varepsilon_{\tilde{R},t} \end{bmatrix}$$

$$\Pi\ \eta_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{E}l,t} \\ \eta_{\mathbb{E}c,t} \\ \eta_{\mathbb{E}z,t} \\ \eta_{\mathbb{E}\tau_x,t} \\ \eta_{\mathbb{E}\tau_b,t} \\ \eta_{\mathbb{E}\pi,t} \end{bmatrix}$$

C.4.3 BCA - Chari et al. (2007) with Adjustment Costs

Allowing for adjustment costs in the standard BCA model requires modifying the state vector $y(t)$ and the matrices $\Gamma_0 y_t$, $\Gamma_1 y_{t-1}$, C, $\Psi\epsilon_t$, $\Pi\eta_t$ in the following way:

$$
\Gamma_0\, y_t = \begin{bmatrix}
\phi^{adj}_{k\mathbb{E}k} & 0 & 0 & 1 & 0 & 0 & 0 & \phi^{adj}_{kx} & \phi_{kl} & 0 & \phi_{kc} & \phi_{k\mathbb{E}l} & \phi_{k\mathbb{E}c} & \phi_{k\mathbb{E}z} & \phi_{k\mathbb{E}\tau_x} & \phi^{adj}_{k\mathbb{E}x} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{yk'} & \phi_{yz} & \phi_{y\tau_l} & 0 & \phi_{yg} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{xk'} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{lk'} & \phi_{lz} & \phi_{l\tau_l} & 0 & \phi_{lg} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \phi_{cg} & 0 & \phi_{cy} & \phi_{cx} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\log \hat{k}_{t+1} \\ \log z_t \\ \tau_{lt} \\ \tau_{xt} \\ \log \hat{g}_t \\ 1 \\ \log \hat{y}_t \\ \log \hat{x}_t \\ \log l_t \\ \log \hat{g}^{obs}_t \\ \log \hat{c}_t \\ \mathbb{E}_t\{\log l_{t+1}\} \\ \mathbb{E}_t\{\log \hat{c}_{t+1}\} \\ \mathbb{E}_t\{\log z_{t+1}\} \\ \mathbb{E}_t\{\tau_{x,t+1}\} \\ \mathbb{E}_t\{\log x_{t+1}\}
\end{bmatrix}
$$

$$
\Gamma_1\, y_{t-1} = \begin{bmatrix}
-\phi^{adj}_{kk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & \rho_{z,\tau_l} & \rho_{z,\tau_x} & \rho_{z,g} & \bar{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_l,z} & \rho_{\tau_l} & \rho_{\tau_l,\tau_x} & \rho_{\tau_l,g} & \bar{\tau}_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_x,z} & \rho_{\tau_x,\tau_l} & \rho_{\tau_x} & \rho_{\tau_x,g} & \bar{\tau}_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{g,z} & \rho_{g,\tau_l} & \rho_{g,\tau_x} & \rho_g & \bar{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{yk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{xk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{lk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\log \hat{k}_t \\ \log z_{t-1} \\ \tau_{lt-1} \\ \tau_{xt-1} \\ \log \hat{g}_{t-1} \\ 1 \\ \log \hat{y}_{t-1} \\ \log \hat{x}_{t-1} \\ \log l_{t-1} \\ \log \hat{g}^{obs}_{t-1} \\ \log \hat{c}_{t-1} \\ \mathbb{E}_{t-1}\{\log l_t\} \\ \mathbb{E}_{t-1}\{\log \hat{c}_t\} \\ \mathbb{E}_{t-1}\{\log z_t\} \\ \mathbb{E}_{t-1}\{\tau_{x,t}\} \\ \mathbb{E}_{t-1}\{\log x_t\}
\end{bmatrix}
$$

$$
C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'
$$

$$\Psi\ \epsilon_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q11 & 0 & 0 & 0 \\ q21 & q22 & 0 & 0 \\ q31 & q32 & q33 & 0 \\ q41 & q42 & q43 & q44 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{z,t} \\ \varepsilon_{\tau_l,t} \\ \varepsilon_{\tau_x,t} \\ \varepsilon_{g,t} \end{bmatrix}$$

$$\Pi\ \eta_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{E}l,t} \\ \eta_{\mathbb{E}c,t} \\ \eta_{\mathbb{E}z,t} \\ \eta_{\mathbb{E}\tau_x,t} \\ \eta_{\mathbb{E}x_t} \end{bmatrix}$$

C.4.4 Monetary BCA - Šustek (2011) with Adjustment Costs

Allowing for adjustment costs in the standard BCA model requires modifying the state vector $y(t)$ and the matrices $\Gamma_0 y_t$, $\Gamma_1 y_{t-1}$, C, $\Psi\epsilon_t$, $\Pi\eta_t$ in the following way:

$$\Gamma_0\, y_t = \begin{bmatrix}
\phi^{adj}_{k\mathbb{E}k} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \phi^{adj}_{kx} & \phi_{kl} & 0 & 0 & 0 & \phi_{kc} & \phi_{k\mathbb{E}l} & \phi_{k\mathbb{E}c} & \phi_{k\mathbb{E}z} & \phi_{k\mathbb{E}\tau_x} & 0 & 0 & \phi^{adj}_{k\mathbb{E}\tau_x} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{yk'} & \phi_{yz} & \phi_{y\tau_l} & 0 & \phi_{yg} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{xk'} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{lk'} & \phi_{lz} & \phi_{l\tau_l} & 0 & \phi_{lg} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \phi_{\pi_0} & \phi_{\pi y} & 0 & 0 & 0 & \phi_{\pi} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \phi_{R\tau_b} & 0 & 0 & 0 & 0 & \phi_{Rl} & 0 & 0 & -1 & \phi_{Rc} & \phi_{R\mathbb{E}l} & \phi_{R\mathbb{E}c} & 0 & 0 & \phi_{R\mathbb{E}\tau_b} & 1 & 0 \\
0 & 0 & 0 & 0 & \phi_{cg} & 0 & 0 & 0 & \phi_{cy} & \phi_{cx} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\log \hat{k}_{t+1} \\ \log z_t \\ \tau_{lt} \\ \tau_{xt} \\ \log \hat{g}_t \\ \tau_{bt} \\ \tilde{R}_t \\ 1 \\ \log \hat{y}_t \\ \log \hat{x}_t \\ \log l_t \\ \log \hat{g}^{obs}_t \\ \pi_t \\ R_t \\ \log \hat{c}_t \\ \mathbb{E}_t\{\log l_{t+1}\} \\ \mathbb{E}_t\{\log \hat{c}_{t+1}\} \\ \mathbb{E}_t\{\log z_{t+1}\} \\ \mathbb{E}_t\{\tau_{x,t+1}\} \\ \mathbb{E}_t\{\tau_{b,t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \\ \mathbb{E}_t\{\log x_{t+1}\}
\end{bmatrix}$$

$$
\Gamma_1\ y_{t-1} = \begin{bmatrix}
-\phi_{kk}^{adj} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & \rho_{z,\tau_l} & \rho_{z,\tau_x} & \rho_{z,g} & \rho_{z,\tau_b} & \rho_{z,\tilde{R}} & \bar{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_l,z} & \rho_{\tau_l} & \rho_{\tau_l,\tau_x} & \rho_{\tau_l,g} & \rho_{\tau_l,\tau_b} & \rho_{\tau_l,\tilde{R}} & \bar{\tau}_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_x,z} & \rho_{\tau_x,\tau_l} & \rho_{\tau_x} & \rho_{\tau_x,g} & \rho_{\tau_x,\tau_b} & \rho_{\tau_x,\tilde{R}} & \bar{\tau}_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{g,z} & \rho_{g,\tau_l} & \rho_{g,\tau_x} & \rho_g & \rho_{g,\tau_b} & \rho_{g,\tilde{R}} & \bar{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tau_b,z} & \rho_{\tau_b,\tau_l} & \rho_{\tau_b,\tau_x} & \rho_{\tau_b,g} & \rho_{\tau_b} & \rho_{\tau_b,\tilde{R}} & \bar{\tau}_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tilde{R},z} & \rho_{\tilde{R},\tau_l} & \rho_{\tilde{R},\tau_x} & \rho_{\tilde{R},g} & \rho_{\tilde{R},\tau_b} & \rho_{\tilde{R}} & \bar{\tilde{R}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{yk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{xk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\phi_{lk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\phi_{\pi R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\log\hat{k}_t \\
\log z_{t-1} \\
\tau_{lt-1} \\
\tau_{xt-1} \\
\log\hat{g}_{t-1} \\
\tau_{bt-1} \\
\tilde{R}_{t-1} \\
1 \\
\log\hat{y}_{t-1} \\
\log\hat{x}_{t-1} \\
\log l_{t-1} \\
\log\hat{g}^{obs}_{t-1} \\
\pi_{t-1} \\
R_{t-1} \\
\log\hat{c}_{t-1} \\
\mathbb{E}_{t-1}\{\log l_t\} \\
\mathbb{E}_{t-1}\{\log\hat{c}_t\} \\
\mathbb{E}_{t-1}\{\log z_t\} \\
\mathbb{E}_{t-1}\{\tau_{x,t}\} \\
\mathbb{E}_{t-1}\{\tau_{b,t}\} \\
\mathbb{E}_{t-1}\{\pi_t\} \\
\mathbb{E}_{t-1}\{\log x_t\}
\end{bmatrix}
$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

$$\Psi \ \epsilon_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ q11 & 0 & 0 & 0 & 0 & 0 \\ q21 & q22 & 0 & 0 & 0 & 0 \\ q31 & q32 & q33 & 0 & 0 & 0 \\ q41 & q42 & q43 & q44 & 0 & 0 \\ q51 & q52 & q53 & q54 & q55 & 0 \\ q61 & q62 & q63 & q64 & q65 & q66 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{z,t} \\ \varepsilon_{\tau_l,t} \\ \varepsilon_{\tau_x,t} \\ \varepsilon_{g,t} \\ \varepsilon_{\tau_b,t} \\ \varepsilon_{\tilde{R},t} \end{bmatrix}$$

$$\Pi \ \eta_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{E}l,t} \\ \eta_{\mathbb{E}c,t} \\ \eta_{\mathbb{E}z,t} \\ \eta_{\mathbb{E}\tau_x,t} \\ \eta_{\mathbb{E}\tau_b,t} \\ \eta_{\mathbb{E}\pi,t} \\ \eta_{\mathbb{E}x,t} \end{bmatrix}$$

D Appendix - Derivatives with Alternative Stepsize

We here report the results for the case where the stepsize used to compute the numerical derivatives is not set to 1e-3 like in Komunjer and Ng (2011) but rather automatically selected by Matlab using the function *nuderst*. The returned step size is the maximum of 1e-4 times the absolute value of the current parameter and 1e-7.

The main results of the previous analysis hold through with two notable exceptions. Fist, as becomes evident from table D-1 and D-3, both the baseline BCA and MBCA model are strictly identifiable already at a tolerance level of 1e-9 (vs. 1e-11). Second, the BCA model is strictly identifiable even when the deep parameters are estimated at a tolerance level of 1e-10, as reported in table D-2. This is not the case for the MBCA model (see Table D-5).

Table D-1: Komunjer and Ng Test Results BCA Model

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	29	25	15	51	40	62	0
e-03	29	25	16	54	45	69	0
e-04	29	25	16	54	45	69	0
e-05	29	25	16	54	45	69	0
e-06	29	25	16	54	45	69	0
e-07	30	25	16	54	46	69	0
e-08	30	25	16	54	46	69	0
e-09	30	25	16	55	46	71	1
e-10	30	25	16	55	46	71	1
e-11	30	25	16	55	46	71	1
Default=2.756906e-12	30	25	16	55	46	71	1
Required	30	25	16	55	46	71	1

Summary: $n_\theta = 30, n_X = 5, n_\varepsilon = 4$.
Order Condition: $n_\theta = 30, n_\delta = 50$.

Table D-2: Komunjer and Ng Test Results BCA Model (Deep Parameters Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	33	25	15	52	44	64	0
e-03	34	25	16	57	50	71	0
e-04	34	25	16	57	50	71	0
e-05	35	25	16	57	51	71	0
e-06	36	25	16	57	52	71	0
e-07	36	25	16	59	52	72	0
e-08	37	25	16	59	53	73	0
e-09	37	25	16	60	53	74	0
e-10	37	25	16	62	53	76	0
e-11	37	25	16	62	53	78	1
Default=5.513812e-12	37	25	16	62	53	78	1
Required	37	25	16	62	53	78	1

Summary: $n_\theta = 37, n_X = 5, n_\varepsilon = 4$.
Order Condition: $n_\theta = 37, n_\delta = 50$.

Table D-3: Komunjer and Ng Test Results MBCA Model

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	60	49	33	101	89	130	0
e-03	60	49	36	108	96	143	0
e-04	60	49	36	109	96	144	0
e-05	60	49	36	109	96	144	0
e-06	60	49	36	109	96	144	0
e-07	61	49	36	109	97	145	0
e-08	61	49	36	110	97	145	0
e-09	61	49	36	110	97	146	1
e-10	61	49	36	110	97	146	1
Default=1.165290e-11	61	49	36	110	97	146	1
e-11	61	49	36	110	97	146	1
Required	61	49	36	110	97	146	1

Summary: $n_\theta = 61, n_X = 7, n_\varepsilon = 6.$

Order Condition: $n_\theta = 61, n_\delta = 105.$

Table D-4: Komunjer and Ng Test Results MBCA Model ($\tau_{b_{ss}}$ and $\tilde{R}_{ss}$ Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	60	49	33	101	89	130	0
e-03	60	49	36	108	96	143	0
e-04	60	49	36	109	96	144	0
e-05	60	49	36	109	96	144	0
e-06	60	49	36	109	96	144	0
e-07	61	49	36	109	97	145	0
Default=1.165290e-11	61	49	36	110	97	146	0
e-08	61	49	36	110	97	146	0
e-09	61	49	36	110	97	146	0
e-10	61	49	36	110	97	146	0
e-11	61	49	36	110	97	146	0
Required	63	49	36	112	99	148	1

Summary: $n_\theta = 63, n_X = 7, n_\varepsilon = 6.$

Order Condition: $n_\theta = 63, n_\delta = 105.$

Problematic Parameters at Tol=1e-3: $\tau_{b_{ss}}$, $\tilde{R}_{ss}$,

Problematic Parameters at Tol=1.000000e-11: $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, $\tau_{b_{ss}}$, $\tilde{R}_{ss}$, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{21}, q_{31}, q_{51}, q_{22}, q_{32}, q_{42}, q_{52}, q_{33}, q_{53}, q_{63}, q_{44}, q_{54},

Table D-5: Komunjer and Ng Test Results MBCA Model (Deep Parameters Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	67	49	35	104	97	133	0
e-03	68	49	36	113	104	146	0
e-04	69	49	36	115	105	148	0
e-05	70	49	36	115	106	148	0
e-06	70	49	36	116	106	149	0
e-07	71	49	36	118	107	149	0
e-08	72	49	36	118	108	151	0
e-09	72	49	36	120	108	154	0
e-10	72	49	36	121	108	156	0
Default=2.330580e-11	72	49	36	121	108	156	0
e-11	72	49	36	121	108	157	1
Required	72	49	36	121	108	157	1

Summary: $n_\theta = 72, n_X = 7, n_\varepsilon = 6$.
Order Condition: $n_\theta = 72, n_\delta = 105$.
Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, $\rho_{\tau_l,z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, $\rho_{\tau_l,g}$, ρ_{τ_l,τ_b}, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, q_{21}, q_{22}, ψ,
Problematic Parameters at Tol=2.330580e-11: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{11}, q_{21}, q_{31}, q_{41}, q_{51}, q_{61}, q_{22}, q_{32}, q_{42}, q_{52}, q_{62}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, q_{65}, q_{66}, g_n, g_z, β, δ, ψ, σ, α, ρ_R, ω_π, ω_y, π_{ss},

Table D-6: Komunjer and Ng Test Results MBCA Model ($\tau_{b_{ss}}$, $\tilde{R}_{ss}$ and Deep Parameters Estimated)

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	Pass
e-02	67	49	35	104	97	133	0
e-03	68	49	36	113	104	146	0
e-04	69	49	36	115	105	148	0
e-05	69	49	36	115	105	148	0
e-06	70	49	36	116	106	148	0
e-07	71	49	36	118	107	150	0
Default=2.330580e-11	72	49	36	121	108	157	0
e-08	72	49	36	119	108	152	0
e-09	72	49	36	121	108	154	0
e-10	72	49	36	121	108	156	0
e-11	72	49	36	121	108	157	0
Required	74	49	36	123	110	159	1

Summary: $n_\theta = 74, n_X = 7, n_\varepsilon = 6$.
Order Condition: $n_\theta = 74, n_\delta = 105$.
Problematic Parameters at Tol=1e-3: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, $\tau_{b_{ss}}$, $\tilde{R}_{ss}$, $\rho_{\tau_l,z}$, ρ_{z,τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, $\rho_{\tau_l,g}$, ρ_{τ_l,τ_b}, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, q_{21}, q_{22}, ψ, σ, ω_π, π_{ss},
Problematic Parameters at Tol=1.000000e-11: z_{ss}, $\tau_{l_{ss}}$, $\tau_{x_{ss}}$, g_{ss}, $\tau_{b_{ss}}$, $\tilde{R}_{ss}$, ρ_z, $\rho_{\tau_l,z}$, $\rho_{\tau_x,z}$, $\rho_{g,z}$, $\rho_{\tau_b,z}$, $\rho_{\tilde{R},z}$, ρ_{z,τ_l}, ρ_{τ_l}, ρ_{τ_x,τ_l}, ρ_{g,τ_l}, ρ_{τ_b,τ_l}, $\rho_{\tilde{R},\tau_l}$, ρ_{z,τ_x}, ρ_{τ_l,τ_x}, ρ_{τ_x}, ρ_{g,τ_x}, ρ_{τ_b,τ_x}, $\rho_{\tilde{R},\tau_x}$, $\rho_{z,g}$, $\rho_{\tau_l,g}$, $\rho_{\tau_x,g}$, ρ_g, $\rho_{\tau_b,g}$, $\rho_{\tilde{R},g}$, ρ_{z,τ_b}, ρ_{τ_l,τ_b}, ρ_{τ_x,τ_b}, ρ_{g,τ_b}, ρ_{τ_b}, $\rho_{\tilde{R},\tau_b}$, $\rho_{z,\tilde{R}}$, $\rho_{\tau_l,\tilde{R}}$, $\rho_{\tau_x,\tilde{R}}$, $\rho_{g,\tilde{R}}$, $\rho_{\tau_b,\tilde{R}}$, $\rho_{\tilde{R}}$, q_{11}, q_{21}, q_{31}, q_{41}, q_{51}, q_{61}, q_{22}, q_{32}, q_{42}, q_{52}, q_{62}, q_{33}, q_{43}, q_{53}, q_{63}, q_{44}, q_{54}, q_{64}, q_{55}, q_{65}, q_{66}, g_n, g_z, β, δ, ψ, σ, α, ρ_R, ω_π, ω_y, π_{ss},